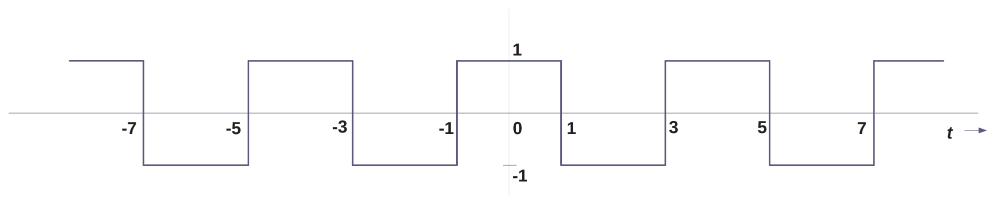
Electrical Engineering and Computer Science ELE 551, Fall 2010
Syracuse University

Solutions to Homework #2

1. [10 points](Adapted from L&D Exercise 2.8-2) For the following periodic signal $g(t)$, find the coefficients a_0, a_n and b_n in the Fourier series representation

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n2\pi f_0 t + b_n \sin n2\pi f_0 t.$$

Hint: (a) $T_0 = 4$; (b) The integral duration can be $[-1, 3]$; (c) Your answer may contain terms such as $\sin\frac{n\pi}{2}$, $\sin\frac{3n\pi}{2}$, etc.



Solution: $T_0 = 4$, and hence $f_0 = 1/4$.

$$a_0 = \frac{1}{T_0}\int_{-1}^{3} g(t)dt = \frac{1}{4}\left[\int_{-1}^{1} 1\cdot dt + \int_{1}^{3}(-1)\cdot dt\right] = 0$$

$$\begin{aligned}
a_n &= \frac{2}{T_0}\int_{-1}^{3} g(t)\cos n2\pi f_0 t dt = \frac{1}{2}\int_{-1}^{3} g(t)\cos\frac{n\pi}{2}t dt \\
&= \frac{1}{2}\left[\int_{-1}^{1}\cos\frac{n\pi}{2}t dt + \int_{1}^{3}(-\cos\frac{n\pi}{2}t)dt\right] \\
&= \frac{1}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi}{2}t\Big|_{-1}^{1} - \frac{2}{n\pi}\sin\frac{n\pi}{2}t\Big|_{1}^{3}\right] \\
&= \frac{1}{2}\left[\frac{4}{n\pi}\sin\frac{n\pi}{2} - \frac{2}{n\pi}\left(\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right)\right] \\
&= \frac{1}{2}\left[\frac{6}{n\pi}\sin\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{3n\pi}{2}\right] \\
&= \frac{1}{n\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right] \qquad \text{you get full credit at this step} \\
&= \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi} & n = 1, 5, 9, \ldots \\ -\frac{4}{n\pi} & n = 3, 7, 11, \ldots \end{cases}
\end{aligned}$$

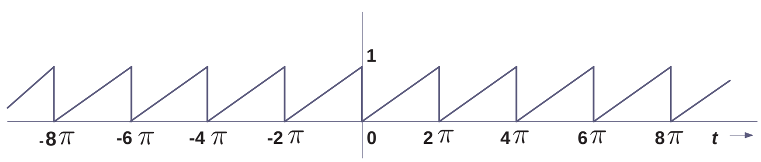
$$
\begin{aligned}
b_n &= \frac{2}{T_0}\int_{-1}^{3} g(t)\sin n2\pi f_0 t dt = \frac{1}{2}\int_{-1}^{3} g(t)\sin\frac{n\pi}{2}t dt \\
&= \frac{1}{2}\left[\int_{-1}^{1}\sin\frac{n\pi}{2}t dt + \int_{1}^{3}(-\sin\frac{n\pi}{2}t)dt\right] \\
&= \frac{1}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi}{2}t\Big|_{-1}^{1} + \frac{2}{n\pi}\cos\frac{n\pi}{2}t\Big|_{1}^{3}\right] \\
&= \frac{1}{2}\left[0 + \frac{2}{n\pi}\left(\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right)\right]
\end{aligned}
$$

It is easy to check that $\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2} = 0$ for both cases when n is odd and even. Hence, $b_n = 0$ for all n.

2. [10 points] (Adapted from L&D Exercise 2.9-1) For the following periodic signal $g(t)$, find the exponential Fourier series, i.e., D_n in the exponential Fourier series representation

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t}.$$

Also find the amplitude and phase of D_n.
Hint: (a) In the duration $0 \le t \le 2\pi$, $g(t) = \frac{1}{2\pi}t$; (b) Separately evaluate D_0 and D_n (with $n \neq 0$); (c) You may want to use $\int te^{at}dt = \frac{e^{at}}{a^2}(at-1)$.



Solutions: $T_0 = 2\pi$, and hence $f_0 = 1/(2\pi)$.

$$D_0 = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-jn2\pi f_0 t}dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}dt = \frac{1}{2}$$

$$
\begin{aligned}
D_n &= \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-jn2\pi f_0 t}dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}e^{-jnt}dt \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi} te^{-jnt}dt = \frac{1}{(2\pi)^2}\frac{e^{-jnt}}{(-jn)^2}(-jnt-1)\Big|_0^{2\pi} \\
&= \frac{1}{(2\pi)^2}\frac{1}{-n^2}(-jn2\pi) = \frac{j}{2\pi n}
\end{aligned}
$$

Hence, the amplitude of D_n is given by

$$|D_n| = \frac{1}{2\pi n}$$

and the phase of D_n is given by

$$\angle D_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ -\frac{\pi}{2} & n < 0 \end{cases}$$