Solutions to Homework #4

1. [10 points](Partially from L&D Exercise 3.8-1) Show that the autocorrelation function of $g(t) = C\cos(2\pi f_0 t + \theta_0)$ is given by $\mathcal{R}_g(\tau) = \frac{C^2}{2}\cos 2\pi f_0 \tau$, and the corresponding PSD is $S_g(f) = \frac{C^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$. (Hint: To compute the autocorrelation function, you may need to use $\lim_{T\to\infty} \frac{\sin(2\pi f_0 T + \theta)}{T} = 0$.)

Solution: Using the definition of the autocorrelation function, we obtain

$$
\begin{aligned}
\mathcal{R}_g(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)g(t+\tau)dt \\
&= \lim_{T\to\infty} \frac{C^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(2\pi f_0 t + \theta_0)\cos(2\pi f_0 t + 2\pi f_0 \tau + \theta_0)dt \\
&\overset{(a)}{=} \lim_{T\to\infty} \left[\frac{C^2}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(2\pi f_0 \tau)dt + \frac{C^2}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(4\pi f_0 t + 2\pi f_0 \tau + 2\theta_0)dt\right] \\
&= \lim_{T\to\infty} \left[\frac{C^2}{2T} T\cos(2\pi f_0 \tau) + \frac{C^2}{2T}\frac{\sin(2\pi f_0(T+\tau) + 2\theta_0) - \sin(2\pi f_0(-T+\tau) + 2\theta_0)}{4\pi f_0}\right] \\
&\overset{(b)}{=} \lim_{T\to\infty} \frac{C^2}{2}\cos(2\pi f_0 \tau) + \lim_{T\to\infty}\frac{C^2}{2T}\frac{\sin(2\pi f_0(T+\tau) + 2\theta_0) - \sin(2\pi f_0(-T+\tau) + 2\theta_0)}{4\pi f_0} \\
&= \frac{C^2}{2}\cos(2\pi f_0 \tau)
\end{aligned}
$$

where step (a) follows because

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

and in step (b), the second term is zero as $T \to \infty$.

The PSD $S_g(f)$ is the Fourier transform of $R_g(\tau)$, and hence from pair 9 of Table 3.1, we have

$$S_g = \frac{C^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

2. [10 points] (Partially from L&D Exercise 3.8-5) Consider a linear system with impulse response $e^{-2t}u(t)$. The linear system input is

$$g(t) = x(t) + w(t)$$

where $x(t)$ is the signal component of input, and $w(t)$ is the noise component of input. Suppose

$$x(t) = \cos\left(6\pi t + \frac{\pi}{3}\right)$$

and the PSD of $w(t)$ is given by

$$S_w(f) = \Pi\left(\frac{f}{4}\right).$$

(a) Find the output power of the signal component due to $x(t)$. (Hint: Use the result from problem 1.)

(b) Find the output power of the noise component due to $w(t)$.

(c) Find the output signal-to-noise ratio (SNR), i.e., ratio of output power due to the signal $x(t)$ and the output power due to the noise signal $w(t)$.

Solution:

(a) From problem 1, the PSD of $x(t)$ is $S_x(f) = |X(f)|^2 = \frac{1}{4}[\delta(f-3) + \delta(f+3)]$. Since the impulse response of the linear system is $h(t) = e^{-2t}u(t)$, its frequency response is given by

$$H(f) = \frac{1}{2 + j2\pi f}.$$

Hence output PSD $S_{y1}(f)$ due to signal input $x(t)$ is

$$\begin{aligned}
S_{y1}(f) = |H(f)|^2 S_x(f) &= \left|\frac{1}{2+j2\pi f}\right|^2 \frac{1}{4}[\delta(f-3) + \delta(f+3)] \\
&= \frac{1}{4 + (2\pi f)^2}\frac{1}{4}[\delta(f-3) + \delta(f+3)] \\
&= \frac{1}{4(4 + (2\pi f)^2)}\delta(f-3) + \frac{1}{4(4 + (2\pi f)^2)}\delta(f+3) \\
&= \frac{1}{4(4 + (6\pi)^2)}\delta(f-3) + \frac{1}{4(4 + (2\pi(-3))^2)}\delta(f+3) \\
&= \frac{1}{16 + 144\pi^2}\delta(f-3) + \frac{1}{16 + 144\pi^2}\delta(f+3)
\end{aligned}$$

The output power due to signal input $x(t)$ is given by

$$\begin{aligned}
P_{y1} &= \int_{-\infty}^{\infty} S_{y1}(f)df = \int_{-\infty}^{\infty} \left[\frac{1}{16 + 144\pi^2}\delta(f-3) + \frac{1}{16 + 144\pi^2}\delta(f+3)\right] df \\
&= \frac{1}{8 + 72\pi^2}
\end{aligned}$$

(b) Since the noise PSD is given, then the output PSD $P_{y2}(f)$ due to noise input $w(t)$ is given by

$$S_{y2}(f) = |H(f)|^2 S_w(f) = \left|\frac{1}{2+j2\pi f}\right|^2 \Pi\left(\frac{f}{4}\right) = \frac{1}{4 + (2\pi f)^2}\Pi\left(\frac{f}{4}\right)$$

The output power due to signal input $w(t)$ is given by

$$\begin{aligned}
P_{y2} &= \int_{-\infty}^{\infty} S_{y2}(f)df = \int_{-\infty}^{\infty} \frac{1}{4 + (2\pi f)^2}\Pi\left(\frac{f}{4}\right) df \\
&= \int_{-2}^{2} \frac{1}{4 + (2\pi f)^2} df \overset{w=2\pi f}{=} \frac{1}{2\pi}\int_{-4\pi}^{4\pi} \frac{1}{4 + w^2} dw \\
&= \frac{1}{2\pi}\frac{1}{2}\tan^{-1}\left(\frac{w}{2}\right)\Big|_{-4\pi}^{4\pi} \\
&= \frac{1}{2\pi}\tan^{-1}(2\pi)
\end{aligned}$$

(c) $SNR = \frac{P_{y1}}{P_{y2}} = \frac{\pi}{(4+36\pi^2)\tan^{-1}(2\pi)}$

3. [10 points] (Partially from L&D Exercise 4.2-1) You are given a baseband signal $m(t) = \cos 1000\pi t$.

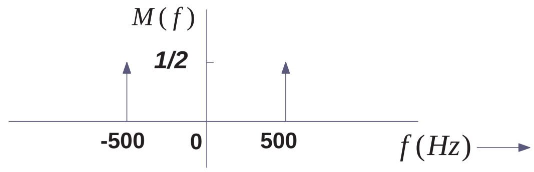
(a) Sketch the spectrum of $m(t)$.

(b) Sketch the spectrum of the DSB-SC signal $m(t)\cos 10,000\pi t$.

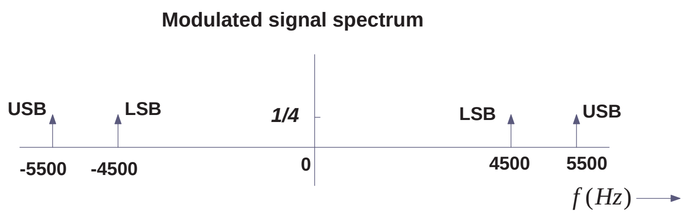
(c) Identify the upper sideband (USB) and the lower sideband (LSB) spectra.

Solutions:

(a) From $m(t)$, we obtain $M(f) = \frac{1}{2}[\delta(f-500)+\delta(f+500)]$. The spectrum includes two impulses located at ± 500 Hz.

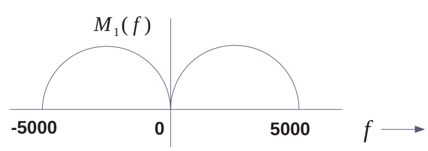


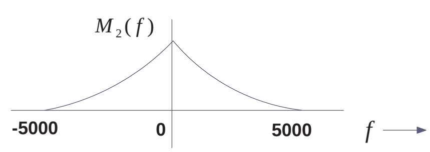
(b) After modulation, the spectrum of $M(f)$ is shifted to the left and right by 5,000 Hz, and suppressed in amplitude by 2. The impulses are now located at 5000 ± 500 Hz and -5000 ± 500 Hz.

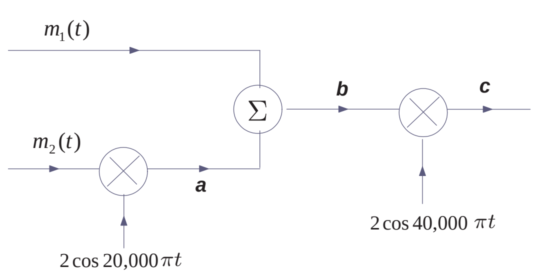


(c) The USB and LSB are marked in the above figure.

4. [10 points] (Partially from L&D Exercise 4.2-7) Two signals $m_1(t)$ and $m_2(t)$, both band-limited to 5000 Hz, are to be transmitted simultaneously over a channel by the multiplexing scheme shown in the figure below. The signal at point b is the multiplexed signal, which now modulates a carrier of frequency $20,000$ Hz. The modulated signal at point c is transmitted over a channel.







(a) Sketch signal spectra at points a, b and c.

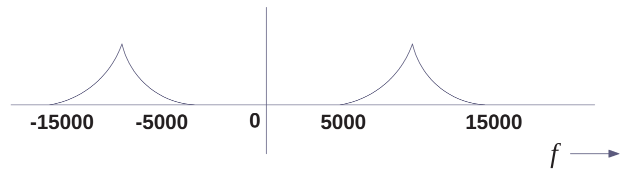
(b) What must be the bandwidth of the distortionless channel? (Hint: The channel must pass the modulated signal entirely so that the signal is not distorted.)

(c) Design a receiver to recover signal $m_1(t)$ from the modulated signal at point c.

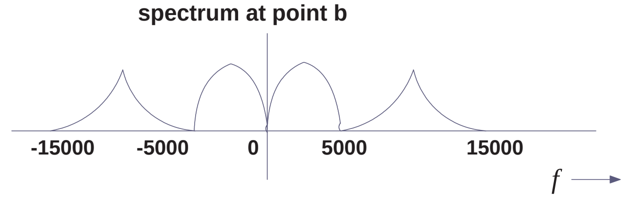
Solutions:

(a) At point a, signal $M_2(f)$ is shifted to the left and right by 10,000 Hz (or 10 KHz). At point b, the spectrum at point (a) adds to spectrum of $M_1(f)$. At point c, the entire spectrum at point b is shifted to the left and right by 20,000 Hz (or 20 KHz).

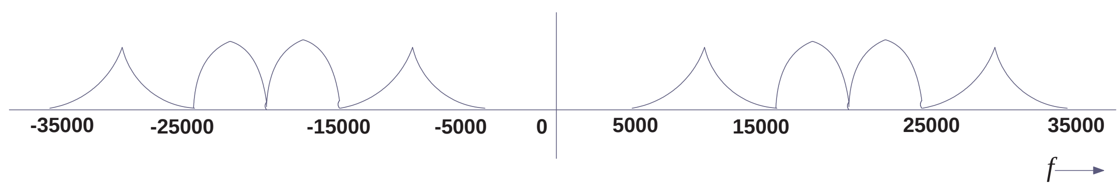
spectrum at point a



spectrum at point b



spectrum at point c



(b) From the spectrum at point c, it is clear that the channel bandwidth must be at least 30,000 Hz (from 5000 Hz to 35000 Hz) in the positive frequency in order not to cut the signal off.

c) The receiver first demodulates $m_1(t)$ by multiplying the signal at point c by $\cos(40000\pi t)$, i.e., the same sinusoidal signal as it is modulated. In this way, the signal $M_1(f)$ is shifted to center at the origin with bandwidth $5000Hz$. Then use low pass filter with bandwidth 5000 Hz to recover $m_1(t)$.