Solutions to Homework #7

1. [10 points] (L&D Exercise 5.2-3) For a message signal

$$m(t) = 2\cos 1000t + 9\cos 2000\pi t$$

(a) Write expressions (do not sketch) for $\varphi_{PM}(t)$ and $\varphi_{FM}(t)$ when $A = 10$, $w_c = 10^6$, $k_f = 1000\pi$, and $k_p = 1$.
(b) Estimate the bandwidths of $\varphi_{FM}(t)$ and $\varphi_{PM}(t)$.

Solution:
(a) $A = 10$, $\omega_c = 10^6$, $k_f = 1000\pi$, $k_p = 1$. Therefore,

$$\begin{aligned}\varphi_{PM}(t) &= A\cos(\omega_c t + k_p m(t)) \\ &= 10\cos(10^6 t + 2\cos 1000t + 9\cos 2000\pi t)\end{aligned}$$

Now, $\int_0^t m(\alpha)d\alpha = 2\sin 1000t/1000 + 9\sin 2000\pi t/(2000\pi)$. Thus,

$$\begin{aligned}\varphi_{FM}(t) &= A\cos\left(\omega_c t + k_f\int_0^t m(\alpha)d\alpha\right) \\ &= 10\cos(10^6 t + 2\pi\sin 1000t + 4.5\sin 2000\pi t)\end{aligned}$$

(b) PM: $\dot{m}(t) = -2000\sin 1000t - 18000\sin 2000\pi t$, so $\dot{m}_p = 2000 + 18000\pi$,

$$\Delta f = \frac{k_p \dot{m}_p}{2\pi} = 1000/\pi + 9000 = 9.32kHz$$

Then from $m(t) = 2\cos 1000t + 9\cos 2000\pi t$, we can see that the highest frequency in $m(t)$ is 1kHz, so $B = 1kHz$. Hence, $B_{PM} = 2(9.32 + 1) = 20.64kHz$.

FM: $m_p = 11, \Delta f = k_f m_p/(2\pi) = 5.5kHz$, and $B_{FM} = 2(\Delta f + B) = 2(5.5 + 1) = 13kHz$.

2. [10 points] (L&D Exercise 5.2-4) An angle-modulated signal with carrier frequency $w_c = 2\pi \times 10^6$ is described by the equation

$$\varphi_{EM}(t) = 10\cos(w_c t + 0.1\sin 2000\pi t)$$

(a) Find the power of the modulated signal.
(b) Find the frequency deviation Δf.
(c) Find the phase deviation $\Delta\phi$.
(d) Estimate the bandwidth of $\varphi_{EM}(t)$.

Solution: From the equation of $\varphi_{EM}(t)$, $a(t) = 0.1 \sin 2000\pi t$. Hence, the message bandwidth is $B = 1kHz$.

(a) $P = 10^2/2 = 50$

(b) $\theta(t) = \omega_c t + 0.1 \sin 2000\pi t$, $\omega_i(t) = \omega_c + 200\pi \cos 2000\pi t$. Hence, $\Delta f = 200\pi/(2\pi) = 100Hz$

(c) $\Delta\phi = 0.1rad$

(d) $B_{EM} = 2(\Delta f + B) = 2(0.1 + 1) = 2.2kHz$.

3. [10 points] (L&D Exercise 5.2-6) Estimate the bandwidth for signals $\varphi_{PM}(t)$ and $\varphi_{FM}(t)$ in problem 2 of Homework 6. Assume the bandwidth of $m(t)$ to be the third-harmonic frequency of $m(t)$.

Solution: In this case, the period of the signal is $T = 10^{-3}$. Hence $w_0 = \frac{2\pi}{T} = 2\pi \times 10^3$ rad, and $f_0 = 10^3$ Hz. We consider the third harmonic frequency as the bandwidth. Hence, $B = 3 \times 10^3 = 3$ kHz.

PM: $k_p = 25$, $\dot{m}_p = \frac{2}{10^{-3}/4} = 8000$, then

$$\Delta f = \frac{k_p \dot{m}_p}{2\pi} = \frac{25 \times 8000}{2\pi} = 31.84kHz$$

So $B_{PM} = 2(\Delta f + B) = 2(31.84 + 3) = 69.68kHz$.
FM: $k_f = 10^5$, $m_p = 1$,

$$\Delta f = \frac{k_f m_p}{2\pi} = 10^5/2\pi = 15.92kHz$$

Thus, $B_{FM} = 2(\Delta f + B) = 2(15.92 + 3) = 37.84kHz$.

4. [10 points] (L&D Exercise 5.2-7) Given $m(t) = \sin 2000\pi t$, $k_f = 200,000\pi$, and $k_p = 10$.
(a) Estimate the bandwidths of $\varphi_{FM}(t)$ and $\varphi_{PM}(t)$.
(b) Repeat part (a) if the message signal amplitude is doubled.
(c) Repeat part (a) if the message signal frequency is doubled.
(d) Comment on the sensitivity of FM and PM bandwidths to the spectrum of $m(t)$.

Solution: From the problem, we know that $B = 2000\pi/(2\pi) = 1kHz$.

(a) PM: $\dot{m}(t) = 2000\pi \cos 2000\pi t$, $\dot{m}_p = 2000\pi$,

$$\Delta f = \tfrac{k_p \dot{m}_p}{2\pi} = 10kHz.$$

So, $B_{PM} = 2(\Delta f + B) = 22kHz$

FM: $m_p = 1$, $\Delta f = k_f m_p/(2\pi) = 100kHz$. Thus, $B_{FM} = 2(\Delta f + B) = 202kHz$.

(b) $m(t) = 2\sin 2000\pi t,\ B = 1kHz.$

PM: $\dot{m}(t) = 4000\pi \cos 2000\pi t,\ \dot{m}_p = 4000\pi;$

$$\Delta f = \frac{k_p \dot{m}_p}{2\pi} = 20kHz.$$

So, $B_{PM} = 2(\Delta f + B) = 42kHz.$

FM: $m_p = 2,\ \Delta f = k_f m_p/(2\pi) = 200kHz.$ Thus, $B_{FM} = 2(\Delta f + B) = 402kHz.$

(c)$m(t) = \sin 4000\pi t,\ B = 2kHz.$

PM: $\dot{m}(t) = 4000\pi \cos 4000\pi t,\ \dot{m}_p = 4000\pi;$

$$\Delta f = \frac{k_p \dot{m}_p}{2\pi} = 20kHz.$$

So, $B_{PM} = 2(\Delta f + B) = 44kHz.$

FM: $m_p = 1,\ \Delta f = k_f m_p/(2\pi) = 100kHz.$ Thus, $B_{FM} = 2(\Delta f + B) = 204kHz.$

(d) Doubling the amplitude of $m(t)$ roughly doubles the bandwidth of both FM and PM. Doubling the frequency of $m(t)$ (i.e. expanding the spectrum of $M(\omega)$ by a factor of 2) has hardly any effect on the FM bandwidth. However, it roughly doubles the bandwidth of PM, indicating that the PM spectrum is sensitive to the shape of the baseband spectrum. The FM spectrum is relatively insensitive to the nature of the spectrum $M(\omega)$.