Electrical Engineering and Computer Science
Syracuse University

Solutions to Homework #9

1. [10 points](L&D Exercise 9.1-7) Consider the random process

$$x(t) = at^2 + b$$

where b is a constant and a is a random variable uniformly distributed in the range $(-2, 2)$, i.e., the pdf of a is given by

$$p_a(x) = \begin{cases} \frac{1}{4} & -2 < a < 2 \\ 0 & otherwise \end{cases}.$$

(a) Find the mean of $x(t)$, i.e., $E[x(t)]$.
(b) Find the autocorrelation function $x(t)$, i.e., $R_x(t_1, t_2) = E[x(t_1)x(t_2)]$.
(c) Determine whether $x(t)$ is a wide-sense stationary process or not.

Solution:
(a)

$$E[x(t)] = E[at^2] + b = E[a] \times t^2 + b = b,$$

where $E[a] = \int_{-2}^{2} \frac{1}{4} x dx = 0$.
(b)

$$E[x(t_1)x(t_2)] = E[(at_1^2 + b)(at_2^2 + b)] = E[a^2 t_1^2 t_2^2 + ab(t_1^2 + t_2^2) + b^2] = E[a^2]t_1^2 t_2^2 + b^2$$

$E[a^2] = \int_{-2}^{2} \frac{1}{4} x^2 dx = \frac{4}{3}$. Hence, $E[x(t_1)x(t_2)] = \frac{4}{3} t_1^2 t_2^2 + b^2$.

(c) $R_x(t_1, t_2)$ is not determined only by $t_1 - t_2$, so it is not a wide-sense stationary process.

2. [10 points] Let $x(t) = kt$, where k is a random variable uniformly distributed over $(-1, 1)$, i.e., the pdf of k is given by

$$p_k(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & otherwise \end{cases}.$$

(a) Find the mean of $x(t)$, i.e., $E[x(t)]$.
(b) Find the autocorrelation function $x(t)$, i.e., $R_x(t_1, t_2) = E[x(t_1)x(t_2)]$.
(c) Determine whether $x(t)$ is a wide-sense stationary process or not.

Solution:
(a)

$$E[x(t)] = E[k]t = 0$$

(b)

$$E[x(t_1)x(t_2)] = E[k^2]t_1 t_2 = \frac{1}{3} t_1 t_2$$

where $E[k^2] = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{3}$.

(c) Since the autocorrelation function $R_x(t_1, t_2)$ is not determined only by $t_1 - t_2$, it is not a wide-sense stationary process.

3. [10 points] Suppose random process $y(t)$ is the output of a linear time-invariant system with WSS input $x(t)$. The transfer function of the linear system is given by $H(f)$. Indicate whether the following statements are true or false. Justify your answers.

(a) The output $y(t)$ is a WSS process.

(b) If $H(f) \leq 1$ for all f, then the power of $y(t)$ is less than or equal to the power of $x(t)$.

Solution:

(a) True.

Proof: Since $X(t)$ is WSS, suppose the mean of $X(t)$ is μ_x and the autocorrelation function of $X(t)$ is $R_x(\tau)$, where $\tau = t_1 - t_2$. We have $Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(t-u)h(u)du$. The mean of $Y(t)$ is given by

$$E[Y(t)] = \int_{-\infty}^{\infty} E[X(t-u)]h(u)du = \mu_x \int_{-\infty}^{\infty} h(u)du,$$

which is not a function of t. The autocorrelation function of $Y(t)$ is:

$$\begin{aligned}
R_y(t_1, t_2) &= E\left(\int_{-\infty}^{\infty} x(t_1-u)h(u)du \int_{\infty}^{\infty} x(t_2-\lambda)h(\lambda)d\lambda\right) \\
&= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} E[x(t_1-u)x(t_2-\lambda)]h(u)h(\lambda)dud\lambda \\
&= \int_{\infty}^{\infty}\int_{\infty}^{\infty} R_x(\tau-u+\lambda)h(u)h(\lambda)dud\lambda
\end{aligned}$$

which depends only on $\tau = t_1 - t_2$, i.e., $R_y(t_1, t_2) = R_y(\tau)$. Hence, $Y(t)$ is also WSS process.

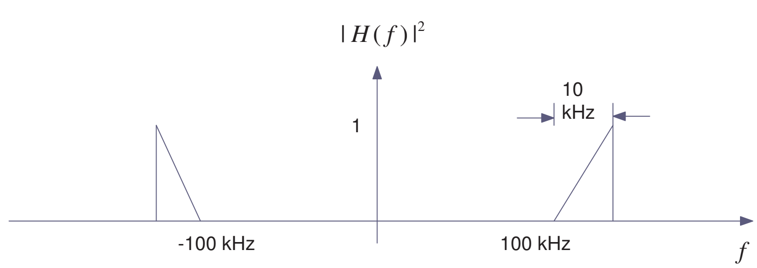
(b) True.

Proof: the power of the input signal is $P_x = \int_{-\infty}^{\infty} S_x(f)df$, and the power of the output signal is $P_y = \int_{-\infty}^{\infty} |H(f)|^2 S_x(f))df \leq \int_{-\infty}^{\infty} S_x(f)df = P_x$. So the output power is smaller than or equal to the input power.

4. [10 points] A white noise process of PSD $\mathcal{N}/2$ is transmitted through a bandpass filter $H(f)$ (shown in the figure below). Assume the center frequency used in this representation is 100 kHz.

(a) Represent the filter output $n(t)$ in terms of quadrature components, and determine $S_{n_c}(f)$ and $S_{n_s}(f)$.

(b) Find $E[n_c^2], E[n_s^2]$, and $E[n^2]$.



Solution:

(a) Representing the filter output in terms of quadrature components:

$$n(t) = n_c(t)\cos(\omega_c t) + n_s(t)\sin(\omega_c t)$$

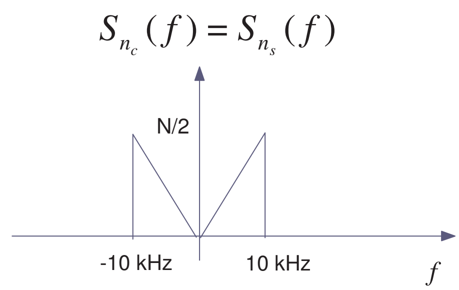
where $\omega_c = 2\pi \times 10^5$. The output PSD is:

$$S_n(f) = |H(f)|^2 S_x(f) = \frac{N}{2}|H(f)|^2$$

and

$$S_{n_c}(f) = S_{n_s}(f) = S_n(f + f_c) + S_n(f - f_c) \quad \text{for } |f| \le f_c$$

which is given in the figure below



(b) $E[n_c^2] = E[n_s^2] = E[n^2] = R_n(0) = \int_{-\infty}^{\infty} S_{n_c}(f)df$, so

$$E[n_c^2] = \frac{N}{2}\left\{\int_{-10K}^{0} -\frac{1}{10K}xdx + \int_{0}^{10K} \frac{1}{10K}xdx\right\} = \frac{N}{2} \cdot 10K = 5000N$$