Solutions to Midterm Exam 1

1. [10pt] True or False. Circle "True" if the statement is correct, and circle "False" if the statement is not correct. Justification is NOT needed.

(a) If $g(t)$ is a periodic signal with period T_0, its average power can be computed as $P_g = \frac{1}{T_0}\int_{T_0} g^2(t)dt$.

True　　　　False

(b) A periodic signal can be expressed by a trigonometric Fourier series, but can NOT be expressed by exponential Fourier series.

True　　　　False

(c) If $g(t)$ is a real function of t, then $G(-f) = G^*(f)$.

True　　　　False

(d) If the LTI system has frequency response $H(f)$. Then energy spectral density (ESD) of its input and output has the relationship $\Psi_y(f) = |H(f)|^2\Psi_x(f)$.

True　　　　False

(e) Consider a LTI system that maps any input $x(t)$ to an output $y(t) = x(t) + x(t-T)$, where T is a delay constant. Then the system response is given by $h(t) = \delta(t) + \delta(t-T)$.

True　　　　False

2. [10pt] Find the Fourier transform of the signal $g(t) = e^{-t}u(t)\sin 2\pi f_c t$, where $u(t)$ is the step function. (Hint: Express $\sin 2\pi f_c t$ in terms of exponentials. You may directly use the Fourier transform for the function $e^{-t}u(t)$, and the frequency shifting property.)

$$g(t) = e^{-t}u(t)\,\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j} = \frac{1}{2j}\left[e^{-t}u(t)\,e^{j2\pi f_c t} - e^{-t}u(t)\,e^{-j2\pi f_c t}\right]$$

$$\Longleftrightarrow \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right] \leftarrow$$

we used ① $e^{-t}u(t) \Leftrightarrow \frac{1}{1+j2\pi f}$

② frequency shifting

$$= \frac{2\pi f_c}{(1+j2\pi f)^2 + (2\pi f_c)^2}$$

3. [10pt] Find the inverse Fourier transform of $G(f)$ given below by Direct Calculation.

$$G(f) = \begin{cases} e^{j\pi/2}, & -B < f < 0 \\ e^{-j\pi/2}, & 0 \le f < B \\ 0, & \text{otherwise} \end{cases}$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df .$$

$$= \int_{-B}^{0} e^{j\frac{\pi}{2}} e^{j2\pi ft} df + \int_{0}^{B} e^{-j\frac{\pi}{2}} e^{j2\pi ft} df .$$

$$= j \frac{e^{j2\pi ft}\Big|_{-B}^{0}}{j2\pi t} - j \frac{e^{j2\pi ft}\Big|_{0}^{B}}{j2\pi t}$$

$$= \frac{1-e^{-j2\pi Bt}}{2\pi t} - \frac{e^{j2\pi Bt}-1}{2\pi t}$$

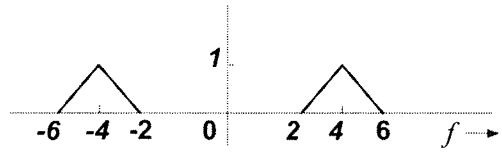
$$= \frac{2-(e^{j2\pi Bt}+e^{-j2\pi Bt})}{2\pi t}$$

$$= \frac{1-\cos 2\pi Bt}{\pi t}$$

4. [10pt] Suppose $G(f)$ is the Fourier transform of $g(t)$.

(a) What is the inverse Fourier transform of $G(f-f_0)+G(f+f_0)$? (Hint: Use frequency shifting property.)

(b) What is the inverse Fourier transform of the following frequency domain representation?



(a) By frequency shifting property:

$$g(t)\, e^{j2\pi f_0 t} \Longleftrightarrow G(f-f_0)$$

$$g(t)\, e^{-j2\pi f_0 t} \Longleftrightarrow G(f+f_0)$$

Hence $g(t)\, e^{j2\pi f_0 t} + g(t)\, e^{-j2\pi f_0 t} \Longleftrightarrow G(f-f_0)+G(f+f_0)$

$= 2g(t)\cos 2\pi f_0 t$.

(b) The function given in the above figure is

$$\Delta\left(\frac{f-4}{4}\right) + \Delta\left(\frac{f+4}{4}\right)$$

Since $B\,\mathrm{sinc}^2(\pi B t) \Longleftrightarrow \Delta\left(\frac{f}{2B}\right)$, then

$2\,\mathrm{sinc}^2(2\pi t) \Longleftrightarrow \Delta\left(\frac{f}{4}\right)$ where $B=2$.

From (a)

$$2\cdot\left(2\,\mathrm{sinc}^2(2\pi t)\right)\cdot \cos 2\pi \cdot \overset{f_0=4}{4} \cdot t \Longleftrightarrow \Delta\left(\frac{f-4}{4}\right)+\Delta\left(\frac{f+4}{4}\right)$$

$\to 4\,\mathrm{sinc}^2(2\pi t)\cos 8\pi t$

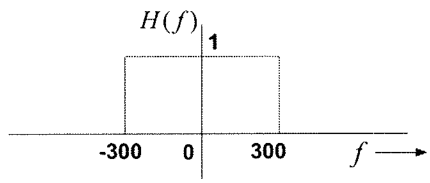
5. [20pt] Consider a baseband signal $m(t) = \cos 500\pi t$. In your plots, please mark the amplitude of each impulse.

(a) Plot the spectrum of $m(t)$.

(b) Plot the spectrum of the DSB-SC signal $2m(t)\cos 4500\pi t$.

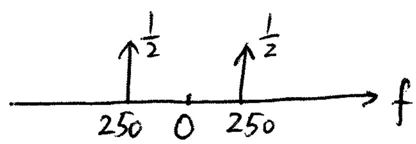
(c) Now suppose the DSB-SC signal in (b) is multiplied by a sinusoidal signal $2\cos 5000\pi t$. Sketch the spectrum now.

(d) Suppose the signal in (c) is passed through the following low pass filter centered at origin with bandwidth 300 Hz. Sketch the spectrum now.

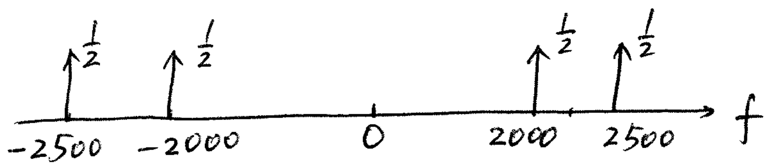


(e) In (d), you do not recover $m(t)$. Why?

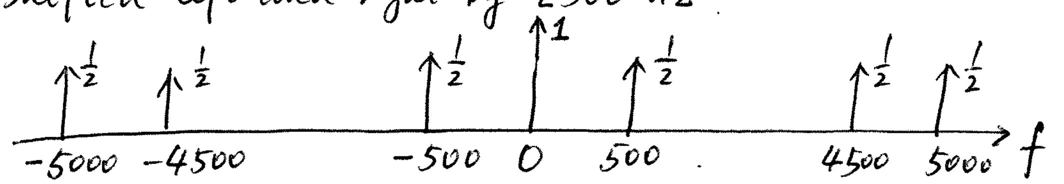
(a)



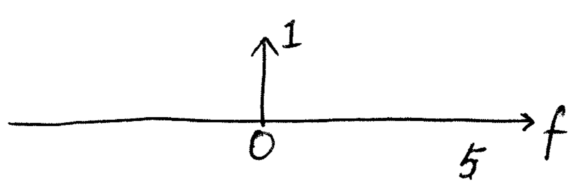
(b) By multifying $m(t)$ with $2\cos 4500\pi t$, the spectrum of $m(t)$ is shifted left and right by 2250 Hz



(c) By multiplying signal in (b) by $2\cos 5000\pi t$, its spectrum is shifted left and right by 2500 Hz



(d)



(e) Not recovering $m(t)$. This is because demodulation uses a frequency 2500 Hz, which is not the carrier frequency for modulation.

6. [20pt] The baseband signal $m(t)$ whose spectrum M(f) is shown in the figure is passed through the system shown in the same figure. The frequency response of $H_1(f)$ and $H_2(f)$ are also given below. Assume that $f_0 >> W$.

(a) Plot the spectrum of $x(t)$.

(b) Show that

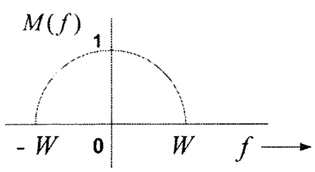
$$y_1(t) = m^2(t) + \frac{1}{2} + \frac{1}{2}\cos 4\pi f_0 t + 2m(t)\cos 2\pi f_0 t.$$

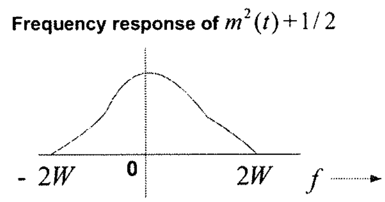
(c) Plot the spectrum of $y_1(t)$. (Hint: Find Fourier transform for each term of $y_1(t)$ given in (b). The spectrum of $m^2(t) + \frac{1}{2}$ is given in the figure below.)

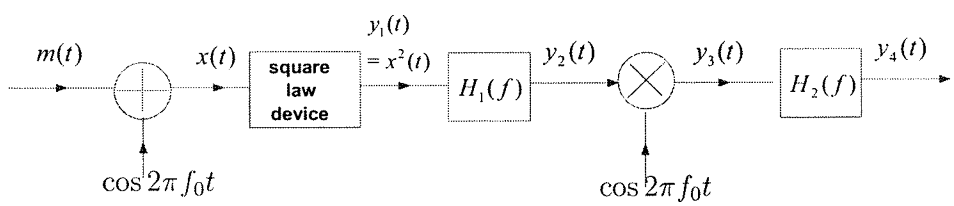
(d) Plot the spectrum of $y_2(t)$.

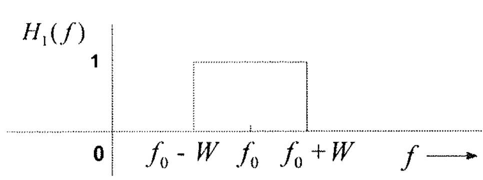
(e) Plot the spectrum of $y_3(t)$.

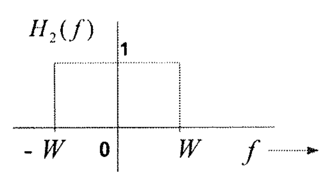
(f) Plot the spectrum of $y_4(t)$.



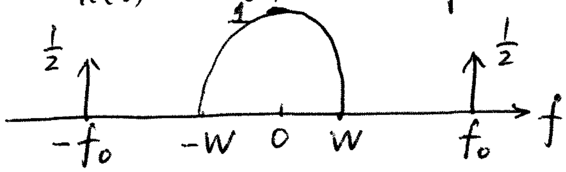








(a) $x(t) = m(t) + \cos 2\pi f_0 t$.

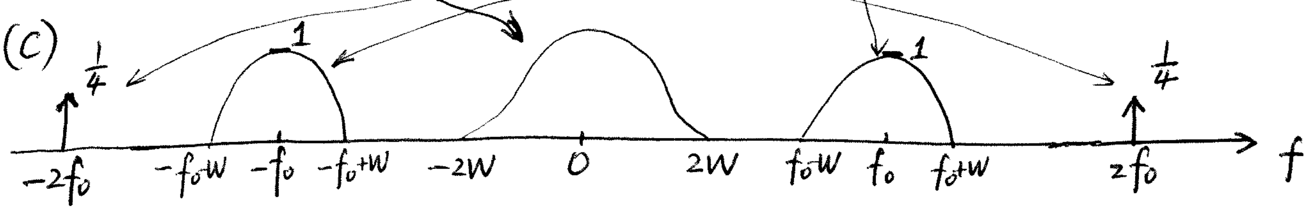


(b) $y_1(t) = x^2(t) = \left(m(t) + \cos 2\pi f_0 t\right)^2$

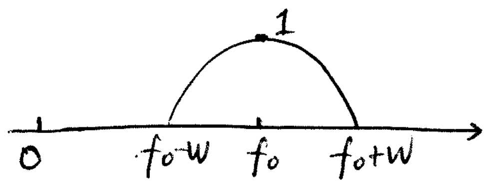
$$= m^2(t) + \cos^2 2\pi f_0 t + 2\, m(t) \cos 2\pi f_0 t$$

$$= \underbrace{m^2(t) + \tfrac{1}{2}} + \tfrac{1}{2}\cos 4\pi f_0 t + 2 m(t) \cos 2\pi f_0 t$$

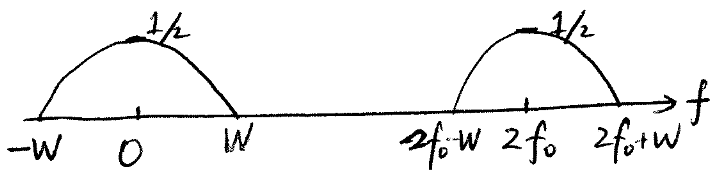
(c)



(d)



(e)



(f)