§5.3. Power Spectral Density (L&Z § 9.3).

- Motivation: convenience of working in frequency domain.
- Recall for deterministic signal

Autocorrelation $R_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) g(t+\tau) dt$.

Power spectral density $S_g(f) = \lim_{T\to\infty} \frac{|G_T(f)|^2}{T}$

$R_g(\tau) \Leftrightarrow S_g(f)$.

- WSS Random processes: average over ensemble of sample paths

$$S_X(f) = \lim_{T\to\infty} \frac{E[|X_T(f)|^2]}{T}$$

$X_T(f)$: Fourier transform of $X(t)$ time truncated $[-T, T]$

- show that $R_X(\tau) \Leftrightarrow S_X(f)$ $R_X(\tau)$ is autocorrelation of $X(t)$

<u>Proof</u>:

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} E[|X_T(f)|^2]$$

$|X_T(f)|^2 = X_T(-f) X_T(f)$.   for real $X(t)$

$$= \int_{-T/2}^{\frac{T}{2}} X(t_1) e^{j2\pi f t_1} \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t_2) e^{-j2\pi f t_2} dt_2.$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t_1) X(t_2) e^{-j2\pi f (t_2 - t_1)} dt_1 dt_2.$$

$$S_X(f) = \lim_{T\to\infty} \frac{E|X_T(f)|^2}{T}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \underbrace{E[X(t_1) X(t_2)]}_{R_X(t_1, t_2) = R_X(\tau)} e^{-j2\pi f (t_2 - t_1)} dt_1 dt_2.$$

let $s = t_1$, $\tau = t_2 - t_1$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} ds \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau.$$

$$= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

Therefore, $S_X(f)$ is the Fourier transform of autocorrelation function.

- Power of a Random Process.

$$P_X = E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df.$$

- For real random process $X(t)$, $R_X(\tau)$ is an even function of τ. The PSD $S_X(f)$ is also real and even function of f.

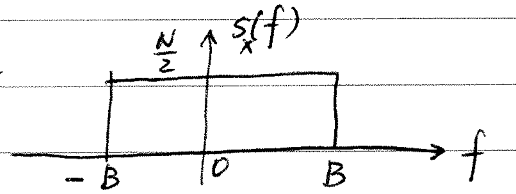
Example. $X(t) = A\cos(\omega_c t + \theta)$, where θ is unif $[0, 2\pi]$

$$R_X(\tau) = \frac{A^2}{2}\cos\omega_c\tau.$$

$$S_X(f) = \frac{A^2}{4}[\delta(f+f_c) + \delta(f-f_c)]$$

$$P_X = R_X(0) = \frac{A^2}{2}$$

Example Given



$$S_X(f) = \frac{N}{2}\Pi\left(\frac{f}{2B}\right).$$

$$R_X(\tau) = NB\,\mathrm{sinc}(2\pi B\tau).$$

$$P_X = R_X(0) = NB.$$

$$\text{or} = \int_{-\infty}^{\infty} S_X(f)\,df = \int_{-B}^{B}\frac{N}{2}\,df = NB.$$

Example DSB-SC $m(t)\cos(\omega_c t + \theta)$; where $m(t)$ is WSS random process and θ is unif $[0, 2\pi]$ and ~~inp~~ independent of $m(t)$

$$x(t) = m(t)\cos(\omega_c t + \theta)$$

$$R_X(\tau) = E[m(t_1)\cos(\omega_c t_1 + \theta)\, m(t_2)\cos(\omega_c t_2 + \theta)]$$

$$= E[m(t_1)\, m(t_2)]\, E[\cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \theta)]$$ ← independency

$$= \frac{1}{2} R_m(\tau)\cos\omega_c\tau$$

$$S_X(f) = \frac{1}{4}\left[S_m(f+f_c) + S_m(f-f_m)\right]$$

$$P_X = R_X(0) = \frac{1}{2} R_m(0) = \frac{1}{2}\overline{m^2(t)}$$

- Cross-correlation of two random processes $X(t)$ and $Y(t)$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

– joint stationary. (in the wide sense)

1. $X(t)$, $Y(t)$ are WSS.
2. $R_{XY}(t_1, t_2) = R_{XY}(\tau)$

– Uncorrelated : if $R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[X(t)]E[Y(t+\tau)] = \mu_X \mu_Y$

– Incoherent or orthogonal

$$R_{XY}(\tau) = 0.$$

- Cross-Power Spectral Density

$$S_{XY}(f) = \lim_{T\to\infty} \frac{E[X_T^*(f)Y_T(f)]}{T}$$

$$R_{XY}(\tau) \Leftrightarrow S_{XY}(f)$$

For real processes $X(t)$ and $Y(t)$:

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

$$S_{XY}(f) = S_{YX}(-f).$$

§5.4. Transmission of Random Processes through Linear Filters (L&Z 9.5)

- $$X(t) \longrightarrow \boxed{H(f)} \longrightarrow Y(t)$$

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau).$$

and $S_Y(f) = |H(f)|^2 S_X(f)$.