Capacity Characterization for State-Dependent Gaussian Channel with a Helper [1]

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Abstract

The state-dependent Gaussian channel with a helper is studied, in which a transmitter communicates with a receiver via a state-corrupted channel. The state is not known to the transmitter nor to the receiver, but known to a helper noncausally, which then wishes to assist the receiver to cancel the state. Differently from previous work that characterized the capacity only in the infinite state power regime, this paper explores the general case with arbitrary state power. A lower bound on the capacity is derived based on an achievable scheme that integrates direct state subtraction and single-bin dirty paper coding. By analyzing this lower bound and further comparing it with the existing upper bounds, the capacity of the channel is characterized for a wide range of channel parameters.

1 Introduction

In this paper, we study the state-dependent channel with a helper (see Figure 1), in which a transmitter wishes to send the message W to a receiver over the state-corrupted channel, and a helper that knows the state information noncausally wishes to assist the receiver to cancel state interference. The state information is not known at the transmitter nor at the receiver. Here, the transmitter that needs to send the message does not know the state, whereas the helper that knows the state does not know the message. Such a mismatched property differentiates this channel from the traditional state-dependent channel studied in [1, 2], where the transmitter knows both the message and the state.

The channel of interest here has been studied previously in [3], in which lower and upper bounds on the capacity were derived with the lower bound based on lattice coding. It was

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further shown that the two bounds match in the asymptotic regime as the state power goes to infinity. Furthermore, a number of more general channel models were studied previously, which include the channel of interest here as a special case. More specifically, in [4, 5], the state-dependent multiple-access channel (MAC) was studied, which can be viewed as the model here with the helper also having its own message to the receiver. Two more general state-dependent MACs were studied in [6] and [7], which can be viewed as the MAC model in [4,5] respectively with the helper further knowing the transmitter's message and with one more state corruption known at the transmitter. In [8], the state-dependent Z-interference channel was studied, which can be viewed as the model here with the helper also having a message to its own receiver. In [9, 10], the state-dependent relay channel was studied, which can be viewed as the model here with the helper also receiving information from the transmitter and serving as a relay. When these models reduce to the model here, the results in [5, 7, 8, 10] characterize the capacity of the Gaussian channel as the state power goes to infinity. In particular, the achievable scheme in [7] is based on lattice coding similar to [3], and the scheme in [5,8,10] can be viewed as single-bin dirty paper coding (i.e., a special case of dirty paper coding [1, 2] with only one bin).

It is clear that in all previous work, the capacity in the regime with finite state power is not characterized. There are two major challenges here: (1) the achievable schemes proposed previously may not be sufficiently good for finite state power regime although they are optimal for infinite state power regime; and (2) the derived lower bounds on the capacity tend to have complicated form to capture correlation of the helper's input and the state, and hence are difficult to analyze and compare with upper bounds which may also have complicated form as in [3, 7].

In this paper, we focus on the finite state power regime, and characterize the capacity for a wide range of channel parameters by matching the derived lower bounds with either the existing upper bound in [3] or the capacity of the channel without state. More specifically, our achievable scheme is based on integration of single-bin dirty paper coding and direct state subtraction (i.e., the helper directly cancels partial state in the received output) with optimal trade-off between the two schemes. Such a scheme is equivalent to the generalized dirty paper coding used in [10] for the state-dependent relay channel, which assumes that the relay input and state are correlated. A lower bound on the capacity is derived based on such a scheme, which takes a complicated form and involves various parameters (i.e., dirty paper parameter and parameter capturing trade-off between two schemes) to be optimized. Our major novelty lies in identifying two special cases to analytically optimize the lower bound so that the optimizing lower bound matches either the upper bound in [3] or the capacity of the channel without state for various channel parameters. We thus establish the capacity under these channel parameters.

Our capacity result can be summarized as follows. If the helper's power is relatively small (compared to the transmitter's power and state power), then the capacity is characterized as a function of the state power, the helper's power and the transmitter's power. In particular, the capacity is strictly less than the capacity of the channel without state, which implies that there exists no achievable scheme that fully cancels the state interference. Here, direct state subtraction is necessary for the achievable scheme to be optimal. On the other hand, if the helper's power is larger than a threshold, then the channel achieves the capacity of the channel without state, which implies that the state can be fully canceled. Here, single-bin dirty paper coding is optimal and direct state subtraction is not necessary. Such characterization of the capacity reduces to the capacity result for infinite state power regime obtained in the previous studies [3, 5, 7, 8, 10].

The rest of the paper is organized as follows. In Section 2, we introduce the channel model. In Section 3, we derive lower bounds on the capacity and characterize the capacity for various channel parameters. In Section 4, we demonstrate our theoretic results via numerical plots. In Section 5, we conclude the paper with several remarks.

2 Channel Model

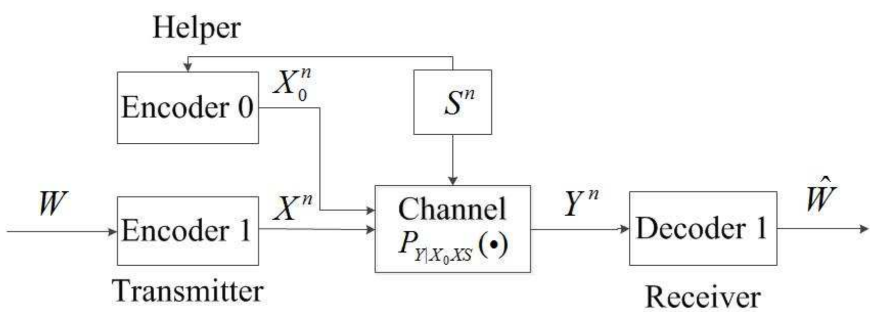

Figure 1: The state-dependent channel with a helper

We study the state-dependent channel with a helper (see Figure 1), in which a transmitter sends a message to a receiver over the state-dependent channel, and a helper that knows the state sequence noncausally wishes to assist the transmission by canceling the state. More specifically, the transmitter has an encoder $f : \mathcal{W} \to \mathcal{X}^n$, which maps a message $w \in \mathcal{W}$ to a codeword $x^n \in \mathcal{X}^n$. The input x^n is transmitted over the channel, which is corrupted by an independent and identically distributed (i.i.d.) state sequence S^n. The state sequence is assumed to be known at neither the transmitter nor the receiver, but at a helper noncausally. Thus, the encoder at the helper, $f_0 : \mathcal{S}^n \to \mathcal{X}_0^n$, maps a state sequence $s^n \in \mathcal{S}^n$ to a codeword $x_0^n \in \mathcal{X}_0^n$ and sends it over the channel. The channel is characterized by the transition probability distribution $P_{Y|X_0,X,S}$. The decoder at the receiver, $g : \mathcal{Y}^n \to \mathcal{W}$, maps a received sequence y^n into a message $\hat{w} \in \mathcal{W}$.

We assume that the message is uniformly distributed over the set $\mathcal{W}$, and define the average probability of error for a length-n code as follows.

$$P_e = \frac{1}{|\mathcal{W}|} \sum_{w=1}^{|\mathcal{W}|} Pr\{\hat{w} \neq w\}. \tag{1}$$

A rate R is *achievable* if there exist a sequence of message sets $\mathcal{W}^{(n)}$ with $|\mathcal{W}^{(n)}| = 2^{nR}$ and encoder-decoder tuples $(f_0^{(n)}, f^{(n)}, g^{(n)})$ such that the average probability of error $P_e^{(n)} \to 0$ as $n \to \infty$. We define the *capacity* of the channel to be supremum of all achievable rates R.

In this paper, we focus on the state-dependent Gaussian channel, with input-output relationship for one channel use given by

$$Y = X_0 + X + S + N \tag{2}$$

where the noise variable N and the state variable S are Gaussian distributed with distributions $N \sim \mathcal{N}(0, 1)$ and $S \sim \mathcal{N}(0, Q)$, and both variables are i.i.d. over channel uses. The channel inputs X_0 and X are subject to the average power constraints

$$\frac{1}{n}\sum_{i=1}^{n} X_{0i}^2 \leqslant P_0 \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \leqslant P. \tag{3}$$

3 Main Results

3.1 Achievable Scheme and Lower Bound

We adopt an achievable scheme that integrates (1) precoding state into a help signal using single-bin dirty paper coding and (2) directly subtracting state. The single-bin dirty paper coding is a special case of dirty paper coding with only one bin, because the bin number corresponds to the message index in dirty paper coding, and here the helper does not know the message to be sent. It has been shown in previous studies that only single-bin dirty paper coding is sufficient to achieve the capacity in the infinite state power regime. This is reasonable because direct state subtraction is not useful when the state power is infinite. However, for the finite state power regime, direct state subtraction can be more efficient and hence should be included in the achievable scheme. In order to achieve the best performance, the achievable scheme should include the two coding schemes with the best trade-off, which results the following achievable rate.

Proposition 1. *For the state-dependent Gaussian channel with a helper, the following rate is achievable*

$$R \leqslant \max_{\substack{(\alpha,\beta)\ s.t.\\ -\sqrt{\frac{P_0}{Q}} \leqslant \beta \leqslant \sqrt{\frac{P_0}{Q}}}} \min\{R_1(\alpha, \beta), R_2(\alpha, \beta)\}, \tag{4}$$

where

$$R_1(\alpha, \beta) = \frac{1}{2}\log\frac{P_0'(P_0' + (1+\beta)^2 Q + P + 1)}{P_0' Q(\alpha - 1 - \beta)^2 + P_0' + \alpha^2 Q}, \tag{5a}$$

$$R_2(\alpha, \beta) = \frac{1}{2}\log\left(1 + \frac{P(P_0' + \alpha^2 Q)}{P_0' Q(\alpha - 1 - \beta)^2 + P_0' + \alpha^2 Q}\right), \tag{5b}$$

and $P_0' = P_0 - \beta^2 Q$.

Proposition 1 is consistent with the achievable rate derived for the state-dependent relay channel in [10] with the noise power on the source to relay link set to infinity. However, the rate expression in [10] is much more complicated due to existence of the relay. It appears to mask the crucial elements required in obtaining the capacity results in the present model. For this reason, we provide a simple proof of achievability that emphasizes the interplay between state subtraction and dirty-paper coding. We would also like to note that the optimality results presented herein (in Section 3.2) are new, and were not known previously to the best of our knowledge.

Proof. We first derive an achievable rate based on single-bin Gel'fand-Pinsker binning scheme for the discrete memoryless state-dependent channel. For a given distribution P_{US}, a number of u^n is generated using the marginal distribution P_U, so that for any s^n, there exists a u^n that is jointly typical with s^n. The helper's input x_0^n is then created based on $P_{X_0|SU}$. The transmitter's input x^n is created based on P_X. The receiver jointly decodes both u^n and x^n. Using such a scheme, the following rate can be shown to be achievable:

$$R \leqslant \min\{I(UX;Y) - I(U;S), I(X;Y|U)\} \tag{6}$$

for some distribution $P_{X_0US}P_XP_{Y|X_0XS}$. The above rate can also be derived from [8, Proposition 2] by setting $X_0' = \phi$.

Proposition 1 then follows by evaluating the mutual information terms in (6) based on the following joint Gaussian distribution for the random variables

$$\begin{aligned} X_0 &= X_0' + \beta S \\ U &= X_0' + \alpha S \end{aligned} \tag{7}$$

where X_0' is independent of S and $X_0' \sim \mathcal{N}(0, P_0')$ with $P_0' = P_0 - \beta^2 Q$ and $-\sqrt{\frac{P_0}{Q}} \leqslant \beta \leqslant \sqrt{\frac{P_0}{Q}}$. □

We note that in (7), the helper's input X_0 contains two parts with X_0' designed using single-bin dirty paper coding, and βS serving for state subtraction. The parameter β captures the trade-off between the two schemes. Furthermore, the achievable rate in (6) can be intuitively understood as follows. The first term

$$I(U,X;Y) - I(U;S) = I(U,X;Y) - I(U,X;S),$$

where (U,X) play the role of the auxiliary variable in Gel'fand-Pinsker scheme. The second term

$$I(X;Y|U) = I(U,X;Y) - I(U;Y)$$

can be interpreted as coding via (U,X) but paying the price needed to convey U to the receiver.

The achievable rate in Proposition 1 is optimized over α and β. The optimization is a max-min problem, i.e., maximization of minimum of $R_1(\alpha,\beta)$ and $R_2(\alpha,\beta)$. In general, such optimization cannot be solved analytically with close-form expressions. In order to obtain further insights of such a lower bound, we consider two special cases in which the optimization is solved analytically and the corresponding achievable rate turns out to achieve the capacity as we present in Section 3.2. The idea is to optimize $R_1(\alpha, \beta)$ and $R_2(\alpha, \beta)$ separately. For example, when $R_1(\alpha, \beta)$ is optimized, if $R_2(\alpha, \beta)$ at the optimizing values of α and β is greater than the optimal $R_1(\alpha, \beta)$, then the corresponding optimal $R_1(\alpha, \beta)$ is achievable. The same argument is applicable to optimizing $R_2(\alpha, \beta)$ instead. Such an idea yields the following two corollaries on the achievable rate.

Corollary 1. *For the state-dependent Gaussian channel with a helper, the following rate R is achievable*

$$R = \max_{-1 \leqslant \rho_{0S} \leqslant 1} \min\{R_1(\rho_{0S}), R_2(\rho_{0S})\} \tag{8}$$

where

$$R_1(\rho_{0S}) = \frac{1}{2}\log\left(1 + \frac{P}{Q + 2\rho_{0S}\sqrt{P_0 Q} + P_0 + 1}\right) + \frac{1}{2}\log(1 + P_0 - \rho_{0S}^2 P_0) \tag{9a}$$

$$R_2(\rho_{0S}) = \frac{1}{2}\log\left(1 + \frac{P((1 + P_0(1 - \rho_{0S}^2))^2 + (1 - \rho_{0S}^2)P_0(\sqrt{Q} + \rho_{0S}\sqrt{P_0})^2)}{(Q + 2\rho_{0S}\sqrt{P_0 Q} + P_0 + 1)(1 + P_0 - \rho_{0S}^2 P_0)}\right). \tag{9b}$$

Proof. It can be shown that $R_1(\alpha, \beta)$ is optimized by $\alpha = \frac{(1+\beta)P_0'}{P_0'+1}$. We further set $\beta = \rho_{0S}\sqrt{\frac{P_0}{Q}}$ to better illustrate the result, where $-1 \leqslant \rho_{0S} \leqslant 1$. Corollary 1 then follows by substituting α and β into (5a) and (5b). □

Corollary 2. *For the state-dependent Gaussian channel with a helper, the following rate R is achievable*

$$R = \min\left\{\frac{1}{2}\log\frac{P_0(P_0 + Q + P + 1)}{P_0 + Q}, \frac{1}{2}\log(1 + P)\right\}. \tag{10}$$

Proof. It can be shown that $R_2(\alpha, \beta)$ is optimized by setting $\alpha = 1$ and $\beta = 0$. Corollary 2 then follows by substituting α and β into (5a) and (5b). □

3.2 Capacity Characterization

In order to characterize the capacity, we first present two useful upper bounds on the capacity. In [3], the following upper bound on the capacity was derived.

Lemma 1. *The capacity of the state-dependent Gaussian channel with a helper is upper bounded as*

$$C \leqslant \max_{-1 \leqslant \rho_{0S} \leqslant 1} \frac{1}{2}\log\left(1 + \frac{P}{Q + 2\rho_{0S}\sqrt{P_0 Q} + P_0 + 1}\right) + \frac{1}{2}\log(1 + P_0 - \rho_{0S}^2 P_0). \tag{11}$$

It is also clear that the capacity of the channel between the transmitter and receiver without state serves as an upper bound on the capacity of the state-dependent channel.

Lemma 2. *The capacity of the state-dependent Gaussian channel with a helper is upper bounded as*

$$C \leqslant \frac{1}{2}\log(1+P). \tag{12}$$

By comparing the achievable rate in Corollary 1 with the upper bound in Lemma 1, we characterize the capacity for various channel parameters in the following theorem.

Theorem 1. *For the state-dependent Gaussian channel with a helper, suppose ρ_{0S}^* maximizes $R_1(\rho_{0S})$ in* (9a)*. If the channel parameters satisfy the following condition:*

$$R_1(\rho_{0S}^*) \leqslant R_2(\rho_{0S}^*), \tag{13}$$

where $R_2(\rho_{0S})$ is given in (9b)*, then the channel capacity $\mathcal{C} = R_1(\rho_{0S}^*)$.*

Proof. Due to Corollary 1 and the condition (13), $R_1(\rho_{0S}^*)$ is achievable. Since such an achievable rate matches the upper bound in Lemma 1, it is thus the capacity of the channel. □

We note that for channels that satisfy the condition (13), the capacity $R_1(\rho_{0S}^*)$ is less than the capacity of the channel without the state. Thus, in such cases, the state interference cannot be fully canceled by any scheme.

Furthermore, by comparing the achievable rate in Corollary 2 with the upper bound in Lemma 2, we further characterize the capacity for an additional set of channel parameters.

Theorem 2. *For the state-dependent Gaussian channel with a helper, if the channel parameters satisfy the following condition:*

$$P_0^2 + P_0Q - Q(P+1) \geqslant 0 \tag{14}$$

then the channel capacity $\mathcal{C} = \frac{1}{2}\log(1+P)$.

Proof. Due to Corollary 2 and the condition (14), the rate $\frac{1}{2}\log(1+P)$ is achievable. Since such an achievable rate matches the upper bound in Lemma 2, it is thus the capacity of the channel. □

It is clear that under the condition (14), the state-dependent channel achieves the capacity of the channel without state. Thus, the state can be fully cancelled even if the state-cognitive node (i.e., the helper) does not know the message.

We further note that as the state power Q goes to infinity, Theorems 1 and 2 collectively characterize the capacity established in the previous studies [3, 5, 7, 8, 10].

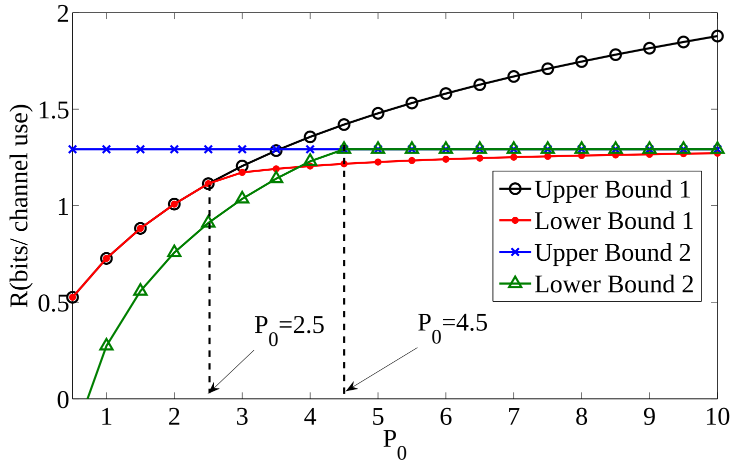

Figure 2: Lower and upper bounds on the capacity for the state-dependent channel with a helper

4 Numerical Result

In this section, we demonstrate our characterization of the capacity via numerical plots.

In Fig. 2, we fix $P = 5$, and $Q = 12$, and plot the lower bounds in Corollaries 1 and 2 and the upper bounds in Lemmas 1 and 2 as functions of the helper's power P_0. It can be seen that the lower bound 1 in Corollary 1 matches the upper bound 1 in Lemma 1 when $P_0 \leq 2.5$, which corresponds to the capacity characterization in Theorem 1, and the lower bound 2 in Corollary 2 matches the upper bound 2 in Lemma 2 when $P_0 \geq 4.5$, which corresponds to the capacity characterization in Theorem 2. The numerical result also suggests that when P_0 is small, the channel capacity is limited by the helper's power and increases as the helper's power P_0 increases. However, as P_0 becomes large enough, the channel capacity is determined only by the transmitter's power P, in which case the state is perfectly canceled. We further note that the channel capacity without state can even be achieved when $P_0 < Q$ (e.g., $4.5 \leq P_0 \leq 10$). This implies that for these cases, the state is fully cancelled not only by state subtraction, but also by precoding the state via single-bin dirty paper coding. We finally note that a better achievable rate can be achieved by the convex envelop of the two lower bounds, which does not yield further capacity result and is not shown in Fig. 2.

In Fig. 3, we fix $P = 5$, and plot the range of the channel parameters (Q, P_0) for which we characterize the capacity. Each point in the figure corresponds to one parameter pair (Q, P_0). The upper shaded area corresponds to channel parameters that satisfy (14), i.e., P_0 is large enough compared to Q, and hence the capacity of the channel without state can be achieved. The lower shaded area corresponds to channel parameters that satisfy (13), and hence the capacity is characterized by a function of not only P, but also P_0 and Q.

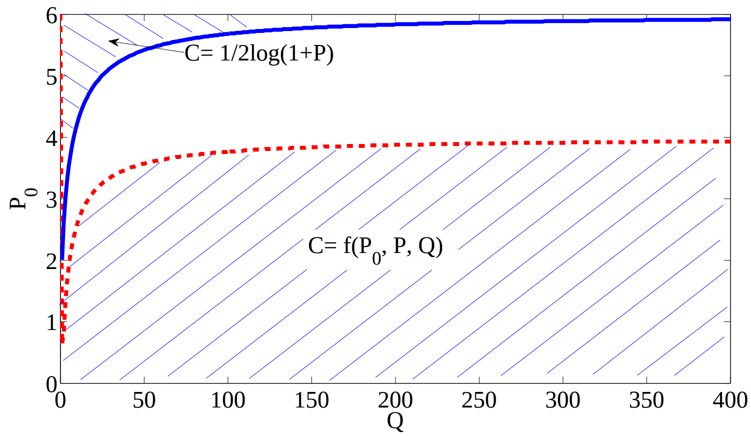

Figure 3: Ranges of parameters for which the capacity is characterized

5 Conclusion

In this paper, we studied the state-dependent point-to-point channel with a helper. Our achievable scheme is based on integration of state subtraction and single-bin dirty paper coding. By analyzing the corresponding lower bound on the capacity, and comparing to the existing upper bounds, we characterize the capacity for various channel parameters. We anticipate that our way of analyzing the lower bound and characterizing the capacity can be applied to characterizing the capacity for other state-dependent networks. We further point out a closely related problem of state masking [11], which has a similar goal of minimizing the impact of the state on the output. It will be interesting to explore if the understanding here can shed any insight on state masking.

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