Fading Multiple Access Relay Channels: Achievable Rates and Opportunistic Scheduling

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***Abstract*—The problem of optimal resource allocation is studied for ergodic fading *orthogonal* multi-access relay channels (MARCs) in which the users (sources) communicate with a destination with the aid of a half-duplex relay that transmits and receives on orthogonal channels. Under the assumption that the instantaneous fading state information is available at all nodes, the maximum sum-rate and the optimal user and relay power allocations (policies) are developed for a decode-and-forward (DF) relay. A known lemma on the sum-rate of two intersecting polymatroids is used to determine the DF sum-rate and the optimal user and relay policies, and to classify fading MARCs into one of three types: (i) *partially clustered* MARCs in which a user is clustered either with the relay or with the destination, (ii) *clustered* MARCs in which all users are either proximal to the relay or to the destination, and (iii) *arbitrarily clustered* MARCs which are a combination of the first two types. Cutset outer bounds are used to show that DF achieves the capacity region for a sub-class of clustered orthogonal MARCs.**

***Index Terms*—Decode-and-forward, ergodic capacity, fading, multiple-access relay channel (MARC), resource allocation.**

I. Introduction

NODE cooperation in multiterminal wireless networks has been shown to improve performance by providing increased robustness to channel variations and by enabling energy savings (see [1]–[7] and the references therein). A specific example of relay cooperation in multiterminal networks is the multi-access relay channel (MARC). The MARC is a network in which several users (source nodes) communicate with a single destination with the aid of a relay [8]. The coding strategies developed for the classical relay channel [9] extend readily to the MARC [10].

We consider a MARC with a half-duplex wireless relay that transmits and receives on two orthogonal channels. Specifically, we model a MARC with a half-duplex relay as an *orthogonal* MARC in which the relay receives on a channel over which all the sources transmit, and transmits to the destination on an orthogonal channel.[1] This channel models a relay-inclusive uplink in a variety of networks such as wireless local area networks (LANs), cellular networks, and sensor networks. The study of wireless relay networks has focused on several performance aspects, including capacity (e.g. [1], [3], [9]), diversity (e.g., [2], [4], [12]), outage (e.g., [13]–[15]), and cooperative coding (e.g., [16], [17]). Equally pertinent is the problem of resource allocation in fading wireless channels in which both source and relay nodes can allocate their transmit powers to enhance a desired performance metric when the fading state information is available. Resource allocation for a variety of relay channels and networks has been studied in several papers, including [5], [13], and [18]–[20]. A common assumption in all these papers is that the source and relay nodes are subject to a total power constraint.

Resource allocation in multi-user relay networks has been studied recently in [21]–[23]. The authors in [21] and [23] consider a specific orthogonal model in which the sources time-duplex their transmissions and are aided in their transmissions by a half-duplex relay, while in [22] the optimal multi-user scheduling policy is determined under the assumption of a nonfading backhaul channel between the relay and destination. In contrast, in this paper, we consider a more general multi-access channel with a half-duplex (orthogonal) relay and model all internode wireless links as ergodic fading channels with perfect channel state information available at all nodes. Assuming a decode-and-forward (DF) relay, we develop the optimal source and relay power allocations and present conditions under which opportunistic time-duplexing of the users is optimal.

The orthogonal MARC is a multi-access generalization of the orthogonal relay channel studied in [6]; however, the optimal DF policies developed in [6] do not extend readily to maximize the DF sum-rate of the MARC. This is because unlike the single-user case, in order to determine the DF sum-rate for the

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[1]Yet another class of orthogonal single-source half-duplex relay channels is defined in [11] in which the source and relay transmit in orthogonal bands. The source transmits in both bands, one of which is received at the relay and the other is received at the destination, such that the relay also transmits in the band received at the destination. In contrast to [11], we assume that all sources transmit in only one of the two orthogonal bands and the relay transmits in the other. Furthermore, we assume that signals in both bands are received at the destination.

MARC, we need to consider the intersection of the two multi-access rate regions that result from decoding at both the relay and the destination. Here, we exploit the polymatroid properties of the two multi-access regions and use a single known lemma on the sum-rate of two intersecting polymatroids [24, chap. 46] to develop inner (DF) and outer bounds on the sum-rate and the rate region. We also specify the sub-class of orthogonal MARCs for which the DF bounds are tight.

A lemma in [24, chap. 46] enables us to classify polymatroid intersections broadly into two sets, namely, the sets of *active* and *inactive cases*. An active or an inactive case result when, in the region of intersection, the constraints on the K-user sum-rate at both receivers are active or inactive, respectively. In the sequel we show that inactive cases suggest *partially clustered* topologies in which a subset of users is clustered closer to one of the receivers while the complementary subset is closer to the remaining receiver. On the other hand, active cases can result from specific *clustered* topologies such as those in which all sources and the relay are clustered or those in which the relay and the destination are clustered, or more generally, from topologies that are either a combination of the two clustered models or of a clustered and a partially clustered model. For both the active and inactive cases, the polymatroid intersection lemma yields closed form expressions for the sum-rates which in turn allows one to develop the sum-rate optimal power allocations (policies).

We first develop the DF sum-rate maximizing power policies for a K-user orthogonal MARC. Using the polymatroid intersection lemma we show that the DF sum-rate averaged over all fading states is achieved by either one of five disjoint cases, two inactive and three active, or by a *boundary case* that lies at the boundary of an active and an inactive case. We develop the sum-rate for all cases and show that the sum-rate maximizing DF power policy either: 1) exploits the multi-user fading diversity to opportunistically schedule users analogously to the fading multiple access channel (MAC) [25], [26] though the optimal multi-user policies are not necessarily water-filling solutions, or 2) involves simultaneous water-filling over two independent point-to-point links.

Using similar techniques, we also develop the K-user DF rate region. Finally, we develop the cutset outer bounds on the sum-capacity. We show that DF achieves the sum-capacity for a class of orthogonal MARCs in which the sources and relay are clustered such that the outer bound on the K-user sum-rate at the destination dominates all other sum-rate outer bounds. We also show that DF achieves the capacity region when the cutset bounds at the destination are the dominant bounds for all rate points on the boundary of the outer bound rate region.

The paper is organized as follows. In Section II, we present the channel models and introduce polymatroids and a lemma on their intersections. In Section III we develop the DF rate region for ergodic fading orthogonal MARCs. In Section IV we develop the power policies that maximize the DF sum-rate for a two-user MARC. In Section IV we extend the analysis to the K-user orthogonal MARC as well as to nonorthogonal models. In Section VI, we present outer bounds and illustrate our results numerically. Finally, in Section VIII, we summarize our contributions.

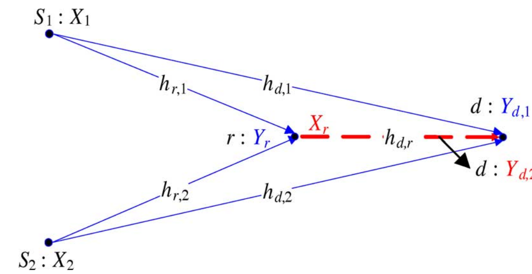

Fig. 1. A two-user orthogonal MARC.

II. Channel Model and Preliminaries

A. Orthogonal Fading MARC

A K-user MARC consists of K source nodes numbered $1, 2, \ldots, K$, a relay node r, and a destination node d. We write $\mathcal{K} = \{1, 2, \ldots, K\}$ to denote the set of sources, $\mathcal{T} = \mathcal{K} \cup \{r\}$ to denote the set of transmitters, and $\mathcal{D} = \{r, d\}$ to denote the set of receivers. In an orthogonal MARC, the sources transmit to the relay and destination on one channel, say channel 1, while the half-duplex relay transmits to the destination on an orthogonal channel 2 as shown in Fig. 1. Thus, a fraction θ of the total bandwidth resource is allocated to channel 1 while the remaining fraction $\bar{\theta} = 1 - \theta$ is allocated to channel 2. In the fraction θ, the source k, for all $k \in \mathcal{K}$, transmits the signal X_k while the relay and the destination receive Y_r and $Y_{d,1}$ respectively. In the fraction $\bar{\theta}$, the relay transmits X_r and the destination receives $Y_{d,2}$ where the sources precede the relay in the transmission order. In each symbol time (channel use), we thus have

$$Y_r = \sum_{k=1}^{K} H_{r,k} X_k + Z_r \tag{1}$$

$$Y_{d,1} = \sum_{k=1}^{K} H_{d,k} X_k + Z_{d,1}, \quad \text{and} \tag{2}$$

$$Y_{d,2} = H_{d,r} X_r + Z_{d,2} \tag{3}$$

where Z_r, $Z_{d,1}$, and $Z_{d,2}$ are circularly symmetric complex Gaussian noise random variables with zero means and unit variances. We write $\underline{H}$ to denote a random vector of fading gains with entries $H_{m,k}$, for all $m \in \mathcal{D}$ and $k \in \mathcal{T}$, $k \neq m$. We use $\underline{h}$ to denote a realization of $\underline{H}$. We assume the fading process $\{\underline{H}\}$ is stationary and ergodic over time but not necessarily Gaussian. Note that the channel gains $H_{m,k}$ are not assumed to be independent, for all m and k. We further assume that the parameter θ is fixed *a priori*, the same for every channel state, and is known at all nodes. As with the classical relay channel, the relay is assumed to be causal, and hence, the signal X_r at the relay in each channel use depends causally only on the Y_r received in the previous channel uses.

Over n uses of the channel, the source and relay transmit sequences $\{X_{k,i}\}$ and $\{X_{r,i}\}$, respectively, which are constrained in power according to

$$\sum_{i=1}^{n} |X_{k,i}|^2 \leq n\bar{P}_k, \quad \text{for all } k \in \mathcal{T}. \tag{4}$$

Since the sources and relay know the fading states of the links on which they transmit, they can allocate their transmitted signal powers according to the channel state information. A power policy $\underline{P}(\underline{h})$ is a mapping from the fading state space consisting of the set of all fading instantiations $\underline{h}$ to the set of positive real values in $\mathcal{R}_+^{K+1}$. The entries of $\underline{P}(\underline{h})$ are $P_k(\underline{h})$, the power policy at user k, for all $k \in \mathcal{T}$. While $\underline{P}(\underline{h})$ denotes the map for a particular fading instantiation, we write $\underline{P}(\underline{H})$ to explicitly describe the policy for the entire set of random channel states. Thus, we use the notation $\underline{P}(\underline{H})$ when averaging over all states or describing a collection of policies, one for every $\underline{h}$. The entries of $\underline{P}(\underline{H})$ are $P_k(\underline{H})$ for all $k \in \mathcal{T}$.

For an ergodic fading channel, (4) then simplifies to

$$\mathbb{E}[P_k(\underline{H})] \leq \bar{P}_k \quad \text{for all } k \in \mathcal{T} \tag{5}$$

where the expectation in (5) is over the distribution of $\underline{H}$. We denote the set of all feasible policies $\underline{P}(\underline{h})$, i.e., the power policies whose entries satisfy (5), by $\mathcal{P}$. Finally, we write $\underline{\bar{P}}$ to denote the vector of average power constraints with entries $\bar{P}_k$, for all $k \in \mathcal{T}$. Throughout the sequel, we also refer interchangeably to the transmit and receive fractions θ and $1-\theta$ as the first and second fractions, respectively.

We assume perfect channel state information (CSI) at the transmitters and receivers and a relatively long transmission time over which all fading states are seen. In practice channel estimation and feedback typically require a slowly varying channel as well as bandwidth and energy resources at the receivers. Despite such practical constraints, our assumption and the ensuing theoretical analysis defines the optimal performance bounds when the fading states are known perfectly at all nodes which in turn can serve as an upper bound on the performance of practical systems. Determining such performance bounds has led to fundamental results on ergodic capacities and optimal policies for many important ergodic channel models such as point-to-point [27], multiple access [25], [26], broadcast [28], and interference channels [29], [30].

Remark 1: We have chosen the bandwidth fraction θ to be fixed *a priori* to make the analysis and elucidation of our results easier; furthermore, such an assumption also models practical networks for which dynamic change of bandwidth fractions may not be straightforward or feasible. In general, however, θ can be chosen to maximize the sum-rate. Our analysis can be extended in a straightforward manner for the case of variable θ, and where possible, we generalize our expressions to allow for this. Later in the sequel, we will illustrate our results for both fixed and varying θ.

Remark 2: An alternate mechanism for half duplexed relay transmissions is to use independent time slots for the users and the relay. Such models have been considered for the MARC in [10] and, in general, for multiterminal relay networks in [21] and [23].

B. Notation

Before proceeding, we summarize the notation used in the sequel.

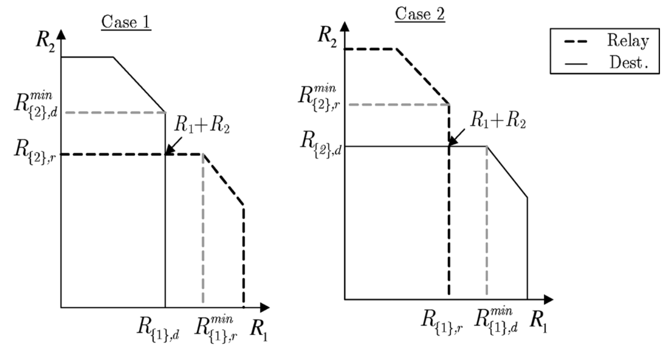

Fig. 2. Rate regions $R_r(\underline{P}(\underline{H}))$ and $R_d(\underline{P}(\underline{H}))$ and sum-rates for cases 1 and 2.

- Random variables (e.g., $H_{k,j}$) are denoted with uppercase letters and their realizations (e.g., $h_{k,j}$) with the corresponding lowercase letters.
- $\mathcal{CN}(0,\mathbf{\Sigma})$ denotes a circularly symmetric complex Gaussian distribution with zero mean and covariance $\mathbf{\Sigma}$.
- $\mathcal{K} = \{1,2,\ldots,K\}$ denotes the set of sources and $\mathcal{T} = \mathcal{K} \cup \{r\}$ denotes the set of all transmitters.
- $\mathbb{E}(\cdot)$ denotes expectation; $C(x)$ denotes $\log(1+x)$ where the logarithm is to the base 2, $(x)^+$ denotes $\max(x,0)$, $I(\cdot;\cdot)$ denotes mutual information, $h(\cdot)$ denotes differential entropy, $\mathcal{X}_\mathcal{S}$ denotes $\{x_k : k \subset \mathcal{S}\}$, and $R_\mathcal{S}$ denotes $\sum_{k\in\mathcal{S}} R_k$ for any $\mathcal{S} \subseteq \mathcal{K}$.
- We use the usual notation for entropy and mutual information [31], [32] and take all logarithms to the base 2 so that our rate units are bits per channel use.
- Rate regions for a fixed θ are denoted with a superscript. $\underline{P}^{DF}(\underline{H})$ and $\underline{P}^{ob}(\underline{H})$ denote the sum-rate optimal power policies for DF and the cutset outer bounds, respectively.

C. Polymatroids

In the sequel, we use the properties of polymatroids to develop the ergodic sum-rate results. Polymatroids have been used to develop capacity characterizations for a variety of multiple-access channel models including the MARC (see for e.g., [26], [33], [34]). We review the following definition of a polymatroid.

Definition 1: Let $\mathcal{K} = \{1,2,\ldots,K\}$ and $f = 2^K \rightarrow \mathfrak{R}_+$ be a set function. The polyhedron

$$\mathcal{B}(f) \equiv \{(R_1,R_2,\ldots,R_K) : R_\mathcal{S} \leq f(\mathcal{S}) \text{ for all } \mathcal{S} \subseteq \mathcal{K},\ R_k \geq 0\} \tag{6}$$

is a polymatroid if $f(\emptyset) = 0$ (normalization), $f(\mathcal{S}) \leq f(\mathcal{P})$ if $\mathcal{S} \subset \mathcal{P}$ (monotonicity), and

$$f(\mathcal{S}) + f(\mathcal{P}) \geq f(\mathcal{S}\cup\mathcal{P}) + f(\mathcal{S}\cap\mathcal{P}) \quad \text{(submodularity)}. \tag{7}$$

We use the following lemma on polymatroid intersections to develop optimal inner and outer bounds on the sum-rate for K-user orthogonal MARCs.

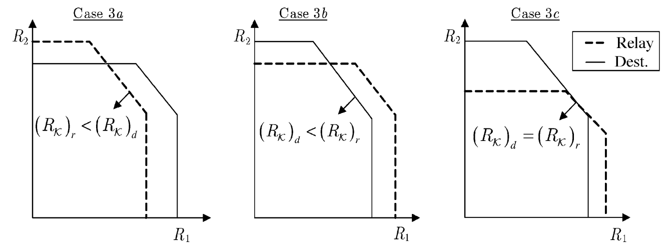

Fig. 3. Rate regions $R_r(\underline{P}(\underline{H}))$ and $R_d(\underline{P}(\underline{H}))$ and sum-rates for cases $3a$, $3b$, and $3c$.

Lemma 1 ([24, p. 796, Cor. 46.1c]): Let $R_{\mathcal{S}} \leq f_1(\mathcal{S})$ and $R_{\mathcal{S}} \leq f_2(\mathcal{S})$, for all $\mathcal{S} \subseteq \mathcal{K}$, be two polymatroids. Then

$$\max R_{\mathcal{K}} = \min_{\mathcal{S} \subseteq \mathcal{K}} (f_1(\mathcal{S}) + f_2(\mathcal{K} \backslash \mathcal{S})). \tag{8}$$

Lemma 1 states that the maximum sum of R_k over all k, denoted by $R_{\mathcal{K}}$, that results from the intersection of two polymatroids, $R_{\mathcal{S}} \leq f_1(\mathcal{S})$ and $R_{\mathcal{S}} \leq f_2(\mathcal{S})$, is given by the minimum of the two K-variable planes $f_1(\mathcal{K})$ and $f_2(\mathcal{K})$ only if both sums are at most as large as the sum of the orthogonal planes $f_1(\mathcal{S})$ and $f_2(\mathcal{K} \backslash \mathcal{S})$, for all $\emptyset \neq \mathcal{S} \subset \mathcal{K}$. We refer to the resulting intersection as belonging to the set of *active cases* (see Fig. 3 for an illustration of the active cases for $K = 2$).

When there exists at least one $\emptyset \neq \mathcal{S} \subset \mathcal{K}$ for which the above condition is not true, an *inactive case* is said to result. For such cases, the maximum K-variable sum in (8) is the sum of two orthogonal rate planes achieved by two complementary subsets of users. As a result, the K-variable sum bounds $f_1(\mathcal{K})$ and $f_2(\mathcal{K})$ are no longer active for this case, and thus, the region of intersection is no longer a polymatroid with $2^K - 1$ faces. For a K-user MARC, there are $2^K - 2$ possible inactive cases. See Fig. 2 for an illustration of the inactive cases for $K = 2$.

The intersection of two polymatroids can also result in a *boundary case* when for any $\mathcal{S} \subset \mathcal{K}$, $f_1(\mathcal{S}) + f_2(\mathcal{K} \backslash \mathcal{S})$ is equal to one or both of the K-user sum-rate planes. The orthogonality of the planes $f_1(\mathcal{S})$ and $f_2(\mathcal{K} \backslash \mathcal{S})$ implies that no two inactive cases have a boundary, and thus, a boundary case arises only between an inactive and an active case. See Figs. 4 and 5 for an illustration of the boundary cases for $K = 2$. Note that by definition, a boundary case is also an active case though for ease of exposition, throughout the sequel we explicitly distinguish between them. From (8), there are three possible active cases corresponding to the three cases in which the sum-rate plane at one of the receivers is smaller than, larger than, or equal to that at the other. In fact, the case in which the sum-rates are equal is also a boundary case between the other two active cases. Thus, there are a total of $(2^K - 1)$ boundary cases for each active case.

In summary, the *inactive set* consists of all intersections for which the constraints on the two sum-rates are not active, i.e., no rate tuple on the sum-rate plane achieved at one of the receivers lies within or on the boundary of the rate region achieved at the other receiver. On the other hand, the intersections for which there exists at least one such rate tuple such that the two sum-rate constraints are active belong to the *active set*. Thus, by definition, the active set also includes those *boundary cases* between the active and inactive cases for which there is exactly one such rate pair.

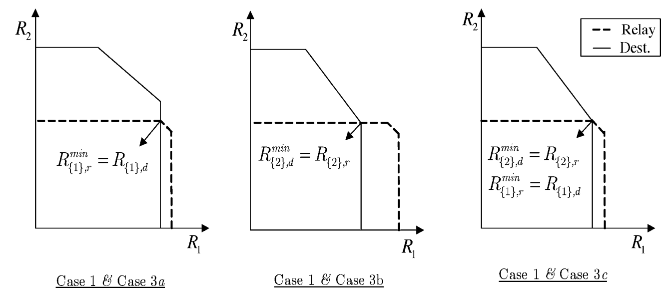

Fig. 4. Rate regions $R_r(\underline{P}(\underline{H}))$ and $R_d(\underline{P}(\underline{H}))$ for cases (1,3a), (1,3b), and (1,3c).

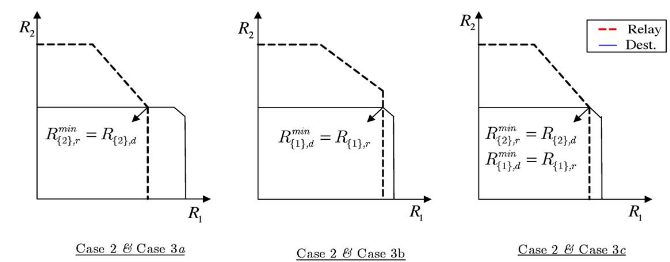

Fig. 5. Rate regions $R_r(\underline{P}(\underline{H}))$ and $R_d(\underline{P}(\underline{H}))$ for cases (2,3a), (2,3b), and (2,3c).

III. Orthogonal MARC: Ergodic DF Rate Region

The DF rate regions for full-duplex discrete memoryless and Gaussian MARCs are developed in [3, Appendix A] (see [34] for a detailed proof) and [35], respectively. The DF rate bounds for the (half-duplex) orthogonal MARC can be obtained from those for the full-duplex MARC by incorporating this restriction via an additional conditioning on a mode random variable that models our orthogonal bandwidth constraint (see [36] for such modeling). In the interest of space, we refer the reader to [34] for the full-duplex bounds and present here directly the DF rate bounds for an orthogonal Gaussian MARC.

For the orthogonal Gaussian MARC with a fixed $\underline{h}$ and θ that are assumed to be known at all nodes, we consider Gaussian signaling at transmitter k with zero mean and variance P_k such that $X_k \sim \mathcal{CN}(0, P_k)$, for all $k \in \mathcal{T}$. Reliable decoding at the relay and at the destination in the appropriate fractions (the relay decodes using signals received in the fraction θ while the destination uses both fractions) requires that the transmitted rates satisfy the multiple access bounds at both receivers. The following proposition summarizes the resulting DF rate region.

Proposition 1: The DF rate region $\mathcal{R}^{\theta}_{DF}(\underline{P})$ for K-user orthogonal Gaussian MARCs with fixed channel states includes the set of all rate pairs $(R_1, R_2, \ldots, R_K)$ that satisfy

$$R_{\mathcal{S}} \leq \min \left\{ \theta C \left(\sum_{k \in \mathcal{S}} |h_{d,k}|^2 P_k \Big/ \theta \right) + \bar{\theta} C \left(|h_{d,r}|^2 P_r / \bar{\theta} \right), \quad \theta C \left(\sum_{k \in \mathcal{S}} |h_{r,k}|^2 P_k \Big/ \theta \right) \right\} \quad \text{for all } \mathcal{S} \subseteq \mathcal{K}. \tag{9}$$

For a stationary and ergodic process $\{\underline{H}\}$, the channel in (1)–(3) can be modeled as a set of parallel Gaussian orthogonal MARCs, one for each fading instantiation $\underline{h}$. For a power policy $\underline{P}(\underline{H})$, assuming Gaussian signaling at the transmitters, the DF rate bounds for this ergodic fading channel are given as a weighted average of the rate bounds achieved in each fading state (the parallel orthogonal Gaussian MARC) where the weights denote the probabilities of occurence of the fading states. Considering the rate regions over all $\underline{P}(\underline{H}) \in \mathcal{P}$ yields the ergodic fading DF rate region, $\mathcal{R}_{DF}^{\theta}(\bar{P})$, where $\bar{P}$ is defined in Section II as a vector of average power constraints at all transmitters (sources and relay). The ergodic fading DF rate region, $\mathcal{R}_{DF}^{\theta}(\bar{P})$, for a fixed bandwidth fraction θ, is summarized by the following theorem.

Theorem 1: The DF rate region $\mathcal{R}_{DF}^{\theta}(\overline{P})$ of a K-user ergodic fading orthogonal Gaussian MARC is

$$\mathcal{R}_{DF}^{\theta}(\overline{P}) = \bigcup_{\underline{P}(\underline{H})\in\mathcal{P}} \left\{\mathcal{R}_r^{\theta}(\underline{P}(\underline{H})) \cap \mathcal{R}_d^{\theta}(\underline{P}(\underline{H}))\right\} \quad (10)$$

where

$$\mathcal{R}_r(\underline{P}(\underline{H})) = \left\{ (R_1, R_2, \ldots, R_K) : R_{\mathcal{S}} \le \mathbb{E}\left[\theta C\left(\sum_{k\in\mathcal{S}} |H_{r,k}|^2 P_k(\underline{H}) \Big/ \theta\right)\right]\right\} \quad \text{for all } \mathcal{S} \subseteq \mathcal{K} \quad (11)$$

and

$$\mathcal{R}_d(\underline{P}(\underline{H})) = \left\{ (R_1, R_2, \ldots, R_K) : R_{\mathcal{S}} \le \mathbb{E}\left[\theta C\left(\sum_{k\in\mathcal{S}} |H_{d,k}|^2 P_k(\underline{H}) \Big/ \theta\right) + \bar{\theta} C\left(|H_{d,r}|^2 P_r(\underline{H})/\bar{\theta}\right)\right], \quad \text{for all } \mathcal{S} \subseteq \mathcal{K} \right\}. \quad (12)$$

Proof: The proof follows from the observation that the channel in (1)–(3) can be modeled as a set of parallel Gaussian orthogonal MARCs, one for each fading instantiation $\underline{h}$ and the fact that independent signals are transmitted in each parallel channel. We use the argument $\underline{P}(\underline{H})$ in denoting the rate regions since the rates are averaged over the channel states. The DF rate region, $\mathcal{R}_{DF}^{\theta}$, is given by the union of such intersections, one for each $\underline{P}(\underline{H}) \in \mathcal{P}$. The convexity of $\mathcal{R}_{DF}^{\theta}$ follows from the convexity of the set $\mathcal{P}$ and the concavity of the log function. ■

Proposition 2: $\mathcal{R}_r^{\theta}(\underline{P}(\underline{H}))$ and $\mathcal{R}_d^{\theta}(\underline{P}(\underline{H}))$ are polymatroids.

Proof: In [34, Sec. IV.B], it is shown that for each choice of the input distribution, the DF rate region is an intersection of two polymatroids, one resulting from the bounds at the relay and the other from the bounds at the destination. For the orthogonal Gaussian MARC, the bounds in (11) and (12) involve a weighted sum of mutual information expressions; using the same approach as in [34, Sec. IV.B], the submodularity of these expressions can be verified in a straightforward manner. ■

Remark 3: The DF rate region is obtained using block Markov encoding at the sources. For the ergodic fading model, the rates in Theorem 1 are obtained assuming that each block is large enough to contain all fading instantiations in an ergodic manner.

Remark 4: For the case where θ can be varied, the DF rate region $\mathcal{R}_{DF}$ is obtained as a union of $\mathcal{R}_{DF}^{\theta}$ over all feasible values of θ, i.e., $\mathcal{R}_{DF} = \bigcup_{\theta\in(0,1)} \mathcal{R}_{DF}^{\theta}$.

In the following sections, we first develop the sum-rate optimal DF power policies for the two-user case and then generalize it for the K-user case.

IV. Two-User Orthogonal MARC: DF Sum-Rate Optimal Power Policy

For ease of notation, throughout the sequel, we write $R_{\mathcal{A},j}$ to denote the sum-rate bound on the users in $\mathcal{A}$ and $R_{\mathcal{A},j}^{\min}$ to denote the sum-rate obtained by successively decoding the users in $\mathcal{A}$ before decoding those in $\mathcal{K}\backslash\mathcal{A}$ at receiver $j = r, d$, i.e., $R_{\mathcal{A},j}^{\min} = R_{\mathcal{K},j} - R_{\mathcal{K}\backslash\mathcal{A},j}$. See Fig. 2 for an illustration. For the two-user case, $R_{\mathcal{K},j}$ and $R_{\mathcal{A},j}$, for all $\mathcal{A} \subset \mathcal{K}$ are given by the sum-rate and single-user bounds in (11) and (12) at the relay and destination, respectively.

The region $\mathcal{R}_{DF}^{\theta}$ in (10) is a union of the intersections of the regions $\mathcal{R}_r^{\theta}(\underline{P}(\underline{H}))$ and $\mathcal{R}_d^{\theta}(\underline{P}(\underline{H}))$ achieved at the relay and destination respectively, where the union is over all $\underline{P}(\underline{H}) \in \mathcal{P}$. Since $\mathcal{R}_{DF}^{\theta}$ is convex, each point on the boundary of $\mathcal{R}_{DF}^{\theta}$ is obtained by maximizing the weighted sum $\mu_1 R_1 + \mu_2 R_2$ over all $\underline{P}(\underline{H}) \in \mathcal{P}$, and for all $\mu_1 > 0$, $\mu_2 > 0$. Specifically, we determine the optimal policy $\underline{P}^*(\underline{H})$ that maximizes the sum-rate $R_1 + R_2$ when $\mu_1 = \mu_2$. Observe from (10) that every point on the boundary of $\mathcal{R}_{DF}^{\theta}$ results from the intersection of the polymatroids (pentagons) $\mathcal{R}_r^{\theta}(\underline{P}(\underline{H}))$ and $\mathcal{R}_d^{\theta}(\underline{P}(\underline{H}))$ for some $\underline{P}(\underline{H})$. In Figs. 2 and 3 we illustrate the five possible choices for the sum-rate resulting from such an intersection for a two-user MARC of which two belong to the inactive set and three to the active set.

The inactive set consists of cases 1 and 2 in which user 1 achieves a significantly larger rate at the relay and destination, respectively, than it does at the other receiver; and vice-versa for user 2. The active set includes cases $3a$, $3b$, and $3c$ shown in Fig. 2 in which the sum-rate at relay r is smaller, larger, or equal, respectively, to that achieved at the destination d. The three boundary cases between case 1 and the three active cases are shown in Fig. 4 while the remaining three between case 2 and the active cases are shown in Fig. 5. We denote a boundary case as case (l, n), $l = 1, 2, n = 3a, 3b, 3c$.

We write $\mathcal{B}_i \subseteq \mathcal{P}$ and $\mathcal{B}_{l,n} \subseteq \mathcal{P}$ to denote the set of power policies that achieve case i, $i = 1, 2, 3a, 3b, 3c$, and case (l, n), $l = 1, 2$, $n = 3a, 3b, 3c$, respectively. We show in the sequel that the optimization is simplified when the conditions for each case are defined such that the sets $\mathcal{B}_i$ and $\mathcal{B}_{l,n}$ are disjoint for all i, l, and n, and thus, are either open or half-open sets such that no two sets share a boundary. Observe that cases 1 and 2 do not share a boundary since such a transition (see Fig. 2) requires passing through case $3a$ or $3b$ or $3c$. Finally, note that Fig. 3 illustrates two specific $\mathcal{R}_r^{\theta}$ and $\mathcal{R}_d^{\theta}$ regions for $3a$, $3b$, and $3c$. For ease of exposition, we write $\mathcal{B}_3 = \mathcal{B}_{3a} \cup \mathcal{B}_{3b} \cup \mathcal{B}_{3c}$.

In general, the occurrence of any one of the disjoint cases depends on both the channel statistics and the policy $\underline{P}(\underline{H})$. Since it is not straightforward to know *a priori* the power allocations that achieve a certain case, we maximize the sum-rate for each case over all allocations in $\mathcal{P}$ and explicitly check whether the optimizing power allocation indeed results in the corresponding case. In the following, we will argue that this can be true for only one case, and the optimizing power policy for this case is the unique solution that achieves the optimal sum-rate.

We write $\underline{P}^{(i)}(\underline{H})$ and $\underline{P}^{(l,n)}(\underline{H})$ to denote the optimal solution for case i and case (l,n), respectively. Explicitly including boundary cases ensures that the sets $\mathcal{B}_i$ and $\mathcal{B}_{l,n}$ are disjoint for all i and (l,n), i.e., these sets are either open or half-open sets such that no two sets share a power policy in common. This in turn simplifies the convex optimization as follows.

Consider case i. The optimal $\underline{P}^{(i)}(\underline{H})$ is first determined by maximizing the sum rate for this case over all $\mathcal{P}$. The resulting sum-rate optimal $\underline{P}^{(i)}(\underline{H})$ must satisfy the conditions for case i, i.e., we require $\underline{P}^{(i)}(\underline{H}) \in \mathcal{B}_i$. If $\underline{P}^{(i)}(\underline{H}) \in \mathcal{B}_i$, the optimality of $\underline{P}^{(i)}(\underline{H})$ follows from the fact that the rate function for each case is strictly concave and that the sets $\mathcal{B}_i$ and $\mathcal{B}_{l,n}$ are disjoint for all i and (l,n) as a result of which $\underline{P}^{(i)}(\underline{H})$ does not maximize the sum-rate for any other case. On the other hand, when $\underline{P}^{(i)}(\underline{H}) \notin \mathcal{B}_i$, we now argue that $R_1 + R_2$ achieves its maximum outside $\mathcal{B}_i$. The proof again follows from the fact that $R_1 + R_2$ for all cases is a strictly concave function of $\underline{P}(\underline{H})$ for all $\underline{P}(\underline{H}) \in \mathcal{P}$. Thus, when $\underline{P}^{(i)}(\underline{H}) \notin \mathcal{B}_i$, for every $\underline{P}(\underline{H}) \in \mathcal{B}_i$ there exists a $\underline{P}'(\underline{H}) \in \mathcal{B}_i$ with a larger sum-rate. Combining this with the fact that the sum-rate expressions are continuous while transitioning from one case to another at the boundary of the open set $\mathcal{B}_i$, ensures that the maximal sum-rate is achieved by some $\underline{P}(\underline{H}) \notin \mathcal{B}_i$. Similar arguments justify maximizing the optimal policy for each case over all $\mathcal{P}$. Due to the strict concavity of the logarithm function, a unique $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$ will satisfy the conditions for its case. The optimal $\underline{P}^{*}(\underline{H})$ is given by this $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$.

The optimization problem for case i or case (l,n) is given by

$$\boxed{\begin{array}{c} S_{DF}^{\theta} = \max\limits_{\underline{P}(\underline{H})\in\mathcal{P}} S_{\theta}^{(i)} \text{ or } \max\limits_{\underline{P}(\underline{H})\in\mathcal{P}} S_{\theta}^{(l,n)} \\ \text{s.t.} \quad \mathbb{E}[P_k(\underline{H})] \le \bar{P}_k, \quad k=1,2,r \\ P_k(\underline{h}) \ge 0, \quad k=1,2,r, \quad \text{for all } \underline{h} \end{array}} \tag{13}$$

where

$$\boxed{\begin{array}{l} S_{\theta}^{(1)} = R_{\{1\},d} + R_{\{2\},r} \\ S_{\theta}^{(2)} = R_{\{1\},r} + R_{\{2\},d} \\ S_{\theta}^{(3a)} = R_{\mathcal{K},r} \\ S_{\theta}^{(3b)} = R_{\mathcal{K},d} \\ S_{\theta}^{(3c)} = (1-\alpha)R_{\mathcal{K},r} + \alpha R_{\mathcal{K},d}, \alpha \text{ s.t. } R_{\mathcal{K},r} = R_{\mathcal{K},d} \\ S_{\theta}^{(l,n)} = (1-\beta)S_{\theta}^{(l)} + \beta S_{\theta}^{(n)}, \ \beta \text{ s.t. } S_{\theta}^{(l)} = S_{\theta}^{(n)} \end{array}} \tag{14}$$

and the subscript θ in $S^{(\cdot)}$ indicates that θ is fixed.

Let S_{MAC} denote the sum-capacity that the two users achieve at the destination in the absence of the relay, i.e., $\theta = 1$ (or $\theta = 0$ and $\bar{P}_r = 0$). From [25] and [26], we know that the optimizing policy $P_{\mathrm{MAC}}(\underline{H})$ simplifies to multi-user opportunistic water-filling. For a fixed θ the maximum achievable sum-rate is then given by

$$\max\left(S_{\mathrm{MAC}}, S_{DF}^{\theta}\right). \tag{15}$$

More generally, when all feasible values of the bandwidth fraction θ are allowed, the maximum achievable sum-rate is given by

$$\max\left(S_{\mathrm{MAC}}, \max_{\theta\in(0,1)} S_{DF}^{\theta}\right). \tag{16}$$

Remark 5: In (16), allowing the range of θ to include $\theta = 1$ covers the MAC without relay case.

Throughout the discussion below, we assume that θ is fixed, and therefore, (15) is used to determine the maximal sum-rate. For the case in which S_{MAC} is larger, it suffices to not allocate any bandwidth resources for relay transmission and simply communicate directly with the destination, i.e., $\theta = 1$. While this may hold for any case, it is particularly possible for cases $3a$, $(1,3a)$, and $(2,3a)$, where the multiple access link to the relay is the bottleneck link. We now determine the sum-rate maximizing policy for each case and assume that (15) is always used to determine the maximal sum-rate.

For each case, we determine the optimal policy using Lagrange multipliers and the *Karush-Kuhn-Tucker* (KKT) conditions [37, 5.5.3]. A detailed analysis is developed in the Appendix and we summarize the KKT conditions and the optimal policies for all cases below. From (14), the KKT conditions for each case x, $x = i, (l,n)$, for all i and (l,n) are given as

$$f_k^{(x)}(\underline{P}(\underline{h})) - \nu_k \ln 2 \le 0, \text{ with equality for } P_k(\underline{h}) > 0 \qquad k = 1,2,r, \quad \text{for all } \underline{h} \tag{17}$$

where ν_k, for all $k = 1,2,r$, are dual variables chosen to satisfy the power constraints in (13) and $f_k^{(x)}(\cdot)$ will be defined later for each case. Specializing the KKT conditions for each case, we obtain the optimal policies for each case as summarized below following which we list the conditions that the optimal policy for each case needs to satisfy.

Case 1: The functions $f_k^{(1)}(\underline{P}(\underline{h}))$, $k = 1,2,r$, in (17) for case 1 are

$$f_k^{(1)}(\underline{P}(\underline{h})) = \frac{|h_{m,k}|^2}{(1 + |h_{m,k}|^2 P_k(\underline{h})/\theta)} \qquad (k,m) = (1,d),(2,r) \tag{18}$$

$$f_r^{(1)}(\underline{P}(\underline{h})) = \frac{|h_{d,r}|^2}{(1 + |h_{d,r}|^2 P_k(\underline{h})/\bar{\theta})}. \tag{19}$$

It is straightforward to verify that these KKT conditions simplify to

$$P_k^{(1)}(\underline{h}) = \left(\frac{\theta}{\nu_k \ln 2} - \frac{\theta}{|h_{m,k}|^2}\right)^+ \qquad (k,m) = (1,d),(2,r) \tag{20}$$

and

$$P_r^{(1)}(\underline{h}) = \left(\frac{\bar{\theta}}{\nu_r \ln 2} - \frac{\bar{\theta}}{|h_{d,r}|^2}\right)^+. \tag{21}$$

Case 2: From (14), since $S^{(2)}$ can be obtained from $S^{(1)}$ by interchanging the user indices 1 and 2, the functions

$f_k^{(2)}(\underline{P}(\underline{h}))$, and hence, the KKT conditions for this case can be obtained by replacing the superscript (1) by (2) and using the pairs $(k,m)=(1,r),(2,d)$ in (18)–(20). The resulting optimal policies are

$$P_r^{(2)}(\underline{h}) = P_r^{(1)}(\underline{h}), \quad \text{for all } \underline{h}, \text{ and}$$
$$P_k^{(2)}(\underline{h}) = \left(\frac{\theta}{\nu_k \ln 2} - \frac{\theta}{|h_{m,k}|^2}\right)^+ \quad (k,m)=(1,r),(2,d). \tag{22}$$

Case 3a: The functions $f_k^{(3a)}(\underline{P}(\underline{h}))$, $k=1,2$, satisfying the KKT conditions in (17) are

$$f_k^{(3a)}(\underline{P}(\underline{h})) = |h_{r,k}|^2 \Bigg/ \left(1 + \sum_{k=1}^{2} |h_{r,k}|^2 P_k(\underline{h}) \Big/ \theta\right) \quad k=1,2. \tag{23}$$

Since this case maximizes the multi-access sum-rate at the relay, the optimal user policies are multi-user opportunistic water-filling solutions given by

$$\frac{|h_{r,1}|^2}{v_1} > \frac{|h_{r,2}|^2}{\nu_2} : P_2^{(3a)} = 0, P_1^{(3a)}(\underline{h}) = \left(\frac{\theta}{\nu_1 \ln 2} - \frac{\theta}{|h_{r,1}|^2}\right)^+$$
$$\frac{|h_{r,1}|^2}{v_1} < \frac{|h_{r,2}|^2}{\nu_2} : P_1^{(3a)}(\underline{h}) = 0, P_2^{(3a)} = \left(\frac{\theta}{\nu_2 \ln 2} - \frac{\theta}{|h_{r,2}|^2}\right)^+$$
$$\frac{|h_{r,1}|^2}{v_1} = \frac{|h_{r,2}|^2}{\nu_2} : P_1^{(3a)}(\underline{h}) = P_2^{(3a)}(\underline{h}) = \left(\frac{\theta/2}{\nu_2 \ln 2} - \frac{\theta/2}{|h_{r,2}|^2}\right)^+ \tag{24}$$

Thus, from (24), we see that the sum-rate $S^{(3a)}$ is maximized when each user exploits knowledge of the channel states to *opportunistically schedule* its transmissions when its fading state is better than that of the other. Finally, while the relay power does not explicitly appear in the optimization, since this case results when the sum-rate at the relay is smaller than that at the destination, choosing the optimal relay policy to maximize the sum-rate at the destination, i.e., $P_r^{(3a)}(\underline{h}) = P_r^{(1)}(\underline{h})$, will ensure the case conditions. However, it is worth noting that forwarding via the relay is desirable for this case only if

$$S_\theta^{(3a)}\left(\underline{P}^{(3a)}(\underline{H})\right) > S_{\text{MAC}}\left(\underline{P}^{(wf)}(\underline{H})\right) \tag{25}$$

is satisfied for the chosen θ (or some θ when θ is allowed to vary). Otherwise, it is better to transmit directly to the destination by setting $\theta=1$, i.e., not use the relay.

Case $3b$*:* The functions $f_k^{(3b)}(\underline{P}(\underline{h}))$, $k=1,2$, satisfying the KKT conditions in (17) can be obtained from (23) by replacing the subscript 'r' by 'd' in (23) while $f_r^{(3b)}(\underline{P}(\underline{h})) = f_r^{(1)}(\underline{P}(\underline{h}))$. Thus, this case maximizes the multi-access sum-rate at the destination and the optimal user policies are multi-user opportunistic water-filling solutions given by

$$\frac{|h_{d,1}|^2}{v_1} > \frac{|h_{d,2}|^2}{\nu_2} : P_2^{(3b)} = 0, P_1^{(3b)}(\underline{h}) = \left(\frac{\theta}{\nu_1 \ln 2} - \frac{\theta}{|h_{d,1}|^2}\right)^+$$
$$\frac{|h_{d,1}|^2}{v_1} < \frac{|h_{d,2}|^2}{\nu_2} : P_1^{(3b)}(\underline{h}) = 0, P_2^{(3b)} = \left(\frac{\theta}{\nu_2 \ln 2} - \frac{\theta}{|h_{d,2}|^2}\right)^+$$
$$\frac{|h_{d,1}|^2}{v_1} = \frac{|h_{d,2}|^2}{\nu_2} : P_1^{(3b)}(\underline{h}) = P_2^{(3b)}(\underline{h}) = \left(\frac{\theta/2}{\nu_2 \ln 2} - \frac{\theta/2}{|h_{d,2}|^2}\right)^+ \tag{26}$$

while the optimal relay policy is a water-filling solution $P_r^{(3b)}(\underline{h}) = P_r^{(1)}(\underline{h})$.

Case $3c$*:* The functions $f_k^{(3c)}(\underline{P}(\underline{h}))$, $k=1,2,r$, satisfying the KKT conditions in (17) are given as

$$f_k^{(3c)}(\underline{P}(\underline{h})) = (1-\alpha) f_k^{(3a)}(\underline{P}(\underline{h})) + \alpha f_k^{(3b)}(\underline{P}(\underline{h})) \quad k=1,2 \tag{27}$$
$$f_r^{(3c)}(\underline{P}(\underline{h})) = \alpha f_r^{(3b)}(\underline{P}(\underline{h})), \quad k=r \tag{28}$$

where the Lagrange multiplier α accounts for the boundary condition

$$R_{\mathcal{K},d}(\underline{P}(\underline{H})) = R_{\mathcal{K},r}(\underline{P}(\underline{H})) \tag{29}$$

and the optimal policy $\underline{P}^{(3c)}(\underline{H}) \in \mathcal{B}_{3c}$ satisfies (29) where $\mathcal{B}_{3c}$ is the set of $\underline{P}(\underline{H})$ that satisfy (29). In the Appendix, using the KKT conditions we show that the optimal user policies are opportunistic in form and are given by

$$\begin{aligned}
&f_1^{(3c)}/\nu_1 > f_2^{(3c)}/\nu_2 : \\
&\quad P_1^{(3c)}(\underline{h}) = \left(\text{root of } F_1^{(3c)}|_{P_2=0}\right)^+ \\
&\quad P_2^{(3c)}(\underline{h}) = 0 \\
&f_1^{(3c)}/\nu_1 < f_2^{(3c)}/\nu_2 : \\
&\quad P_1^{(3c)}(\underline{h}) = 0 \\
&\quad P_2^{(3c)}(\underline{h}) = \left(\text{root of } F_2^{(3c)}|_{P_1=0}\right)^+ \\
&f_1^{(3c)}/\nu_1 = f_2^{(3c)}/\nu_2 : \\
&\quad P_1^{(3c)}(\underline{h}) \text{ and } P_2^{(3c)}(\underline{h}) \text{ obtained} \\
&\quad \text{using iterative non-water-filling}
\end{aligned} \tag{30}$$

where we write

$$F_k^{(3c)} = f_k^{(3c)} - \nu_k \ln 2 \quad k=1,2. \tag{31}$$

Analogous to cases $3a$ and $3b$, the scheduling conditions in (30) depend on both the channel states and the water-filling levels ν_k at both users. However, the conditions in (30) also depend on the power policies, and thus, the optimal solutions are no longer water-filling solutions. In the Appendix we show that the optimal user policies can be computed using an *iterative non-water-filling algorithm* which starts by fixing the power policy of one user, computing that of the other, and vice-versa until the policies converge to the optimal policy. The iterative algorithm is computed for increasing values of $\alpha \in (0,1)$ until the optimal policy satisfies (29) at the optimal α^*. The proof of convergence is detailed in the Appendix. Finally, since $f_r^{(3c)} = \alpha f_r^{(1)}$, the relay's optimal policy simplifies to the water-filling solution given by

$$P_r^{(3c)}(\underline{h}) = \left(\frac{\alpha^* \bar{\theta}}{\nu_r \ln 2} - \frac{\bar{\theta}}{|h_{d,r}|^2} \right)^+ . \quad (32)$$

Boundary Cases (l,n) : A boundary case (l,n) results when

$$S^{(l)} = S^{(n)} \quad l = 1,2, \text{ and } n = 3a, 3b, 3c. \quad (33)$$

Recall that $S^{(l)}$ and $S^{(n)}$ are sum-rates for an inactive case l, and an active case n, respectively. Thus, in addition to the constraints in (13), the maximization problem for these cases includes the additional constraint in (33). For all except the two cases where $n = 3c$, the equality condition in (29) is represented by a Lagrange multiplier α. The two cases with $n = 3c$ have two Lagrange multipliers α_1 and α_2 to also account for both the equality condition in (29) and the condition $S^{(3a)} = S^{(3b)}$.

For the different boundary cases, the functions $f_k^{(l,n)}(\underline{P}(\underline{h}))$, $k = 1,2$, satisfying the KKT conditions in (17) are given as

$$f_k^{(l,n)}(\underline{P}(\underline{h})) = (1-\alpha) f_k^{(l)}(\underline{P}(\underline{h})) + \alpha f_k^{(n)}(\underline{P}(\underline{h})) \quad k = 1,2, n \neq 3c \quad (34)$$

$$f_k^{(l,3c)}(\underline{P}(\underline{h})) = (1-\alpha_1-\alpha_2) f_k^{(l)}(\underline{P}(\underline{h})) + \alpha_2 f_k^{(3a)}(\underline{P}(\underline{h})) + \alpha_1 f_k^{(3b)}(\underline{P}(\underline{h})) \quad k = 1,2 \quad (35)$$

$$f_r^{(l,n)}(\underline{P}(\underline{h})) = (1-\alpha) f_r^{(l)}(\underline{P}(\underline{h})), \quad n = 3a \quad (36)$$

$$f_r^{(l,n)}(\underline{P}(\underline{h})) = (1-\alpha) f_r^{(l)}(\underline{P}(\underline{h})) + \alpha f_r^{(n)}(\underline{P}(\underline{h})) \quad n = 3b \quad (37)$$

$$f_r^{(l,n)}(\underline{P}(\underline{h})) = (1-\alpha_1-\alpha_2) f_r^{(l)}(\underline{P}(\underline{h})) + \alpha_1 f_r^{(3b)}(\underline{P}(\underline{h})) \quad n = 3c. \quad (38)$$

For ease of exposition and brevity, we summarize the KKT conditions and the optimal policies for case $(1,3a)$. In the Appendix, using the KKT conditions we show that the optimal user policies $P_k^{(1,3a)}(\underline{h})$ are opportunistic in form and are given by

$$\begin{aligned}
&\frac{f_1^{(1,3a)}}{\nu_1} > \frac{f_2^{(1,3a)}}{\nu_2} : \\
&\quad P_1^{(1,3a)}(\underline{h}) = \left(\text{root of } F_1^{(1,3a)}|_{P_2=0} \right)^+ \\
&\quad P_2(\underline{h}) = 0 \\
&\frac{f_1^{(1,3a)}}{\nu_1} < \frac{f_2^{(1,3a)}}{\nu_2} : \\
&\quad P_1^{(1,3a)}(\underline{h}) = 0 \\
&\quad P_2(\underline{h}) = \left(\text{root of } F_2^{(1,3a)}|_{P_1=0} \right)^+ \\
&\frac{f_1^{(1,3a)}}{\nu_1} = \frac{f_2^{(1,3a)}}{\nu_2} : \\
&\quad P_1^{(1,3a)}(\underline{h}) \text{ and } P_2(\underline{h}) \text{ solved jointly} \\
&\quad \text{using iterative non-water-filling}
\end{aligned} \quad (39)$$

where $F_k^{(1,3a)} = f_k^{(1,3a)} - \nu_k \ln 2$, for $k = 1,2$. As in case $3c$, the optimal policies take an opportunistic nonwater-filling form and in fact can be obtained by an *iterative nonwater-filling algorithm* as described for case $3c$. Furthermore, analogously to case $3c$, the user policies are computed for increasing values of $\alpha \in (0,1)$ until the optimal policy satisfies (33) at the optimal α^*. The optimal $P_r^{(1,3a)}(\underline{h}) = \alpha^* P_r^{(1)}(\underline{h})$ is a water-filling solution.

The optimal policies for all other boundary cases can be obtained similarly as detailed in the Appendix and can be computed using the iterative algorithm detailed in the Appendix. Specifically, for cases $(l,3c)$, $l = 1,2$, the iterative algorithm is computed for increasing values of $\alpha_1, \alpha_2 \in (0,1)$ until the optimal policy satisfies (33) and (29) at the optimal α_1^* and α_2^*, respectively. For all boundary cases, the optimal user policies are opportunistic nonwater-filling solutions while those for the relay are water-filling solutions. Finally, the sum-rate maximizing policy for any case is the optimal policy only if it satisfies the conditions for that case. The conditions for the cases are

$$\underline{\text{Case 1}} : R_{\{1\},d} < R_{\{1\},r}^{\min} \text{ and } R_{\{2\},r} < R_{\{2\},d}^{\min} \quad (40)$$

$$\underline{\text{Case 2}} : R_{\{1\},r} < R_{\{1\},d}^{\min} \text{ and } R_{\{2\},d} < R_{\{2\},r}^{\min} \quad (41)$$

$$\underline{\text{Case } 3a} : R_{\mathcal{K},r} < R_{\mathcal{K},d},\ R_{\{1\},r}^{\min} < R_{\{1\},d} \text{ and } R_{\{2\},r}^{\min} < R_{\{2\},d} \quad (42)$$

$$\underline{\text{Case } 3b} : R_{\mathcal{K},r} > R_{\mathcal{K},d},\ R_{\{2\},d}^{\min} < R_{\{2\},r} \text{ and } R_{\{1\},d}^{\min} > R_{\{1\},r} \quad (43)$$

$$\underline{\text{Case } 3c} : R_{\{1\},r}^{\min} < R_{\{1\},d}, R_{\{2\},r}^{\min} < R_{\{2\},d}\ R_{\{2\},d}^{\min} < R_{\{2\},r}, \text{ and } R_{\{1\},d}^{\min} > R_{\{1\},r} \quad (44)$$

$$\underline{\text{Case } (1,3a)} : R_{\mathcal{K},r} < R_{\mathcal{K},d} \text{ and } R_{\{2\},r}^{\min} < R_{\{2\},d} \quad (45)$$

$$\underline{\text{Case } (2,3a)} : R_{\mathcal{K},r} < R_{\mathcal{K},d} \text{ and } R_{\{1\},r}^{\min} < R_{\{1\},d} \quad (46)$$

$$\underline{\text{Case } (1,3b)} : R_{\mathcal{K},r} > R_{\mathcal{K},d} \text{ and } R_{\{1\},d}^{\min} < R_{\{1\},r} \quad (47)$$

$$\underline{\text{Case } (2,3b)} : R_{\mathcal{K},r} > R_{\mathcal{K},d} \text{ and } R_{\{2\},d}^{\min} < R_{\{2\},r} \quad (48)$$

$$\underline{\text{Case } (1,3c)} : R_{\mathcal{K},r} = R_{\mathcal{K},d} = R_{\{1\},d} + R_{\{2\},r} < R_{\{1\},r} + R_{\{2\},d} \quad (49)$$

$$\underline{\text{Case } (2,3c)} : R_{\mathcal{K},r} = R_{\mathcal{K},d} = R_{\{1\},r} + R_{\{2\},d} < R_{\{1\},d} + R_{\{2\},r} \quad (50)$$

where in fading state $\underline{h}$, (40)–(50) are evaluated for $X_k \sim \mathcal{CN}(0, P_k^{(x)}(\underline{h})/\theta)$, $k = 1,2$, and $X_r \sim \mathcal{CN}(0, P_r^{(x)}(\underline{h})/\bar{\theta})$ for $x = i, (l,n)$.

The following theorem summarizes the form of $\underline{P}^{DF}$ and presents an algorithm to compute it.

Theorem 2: The optimal policy $\underline{P}^{DF}(\underline{H})$ maximizing the DF sum-rate of a two-user ergodic fading orthogonal MARC is obtained by computing $\underline{P}^{(i)}(\underline{H})$ and $\underline{P}^{(l,n)}(\underline{H})$ starting with the inactive cases 1 and 2, followed by the active cases $3a, 3b$, and $3c$, in that order, and finally the boundary cases (l,n), in the order that cases $(l,3c)$ are the last to be optimized, until for some case the corresponding $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$ satisfies the case conditions. The optimal $\underline{P}^{DF}(\underline{H})$ is given by the optimal $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$ that satisfies its case conditions and falls into one of the following three categories:

Inactive Cases: The optimal policy for the two users is such that one user water-fills over its link to the relay while the other water-fills over its link to the destination. The optimal relay policy $P_r^{DF}(\underline{H})$ is water-filling over its direct link to the destination.

Cases $(3a, 3b, 3c)$*:* The optimal user policy $P_k^{DF}(\underline{H})$, for all $k \in \mathcal{K}$, is opportunistic water-filling over its link to the relay for case $3a$ and to the destination for case $3b$. For case $3c$, $P_k^{DF}(\underline{H})$, for all $k \in \mathcal{K}$, takes an opportunistic nonwater-filling form and depends on the channel gains of user k at both receivers. The optimal relay policy $P_r^{DF}(\underline{h})$ is water-filling over its direct link to the destination.

Boundary Cases: The optimal user policy $P_k^{DF}(\underline{H})$, for all $k \in \mathcal{K}$, takes an opportunistic nonwater-filling form. The optimal relay policy $P_r^{DF}(\underline{H})$ is water-filling over its direct link to the destination.

Proof: The closed form expressions for the optimal policies for each case are developed in the Appendix. The need for an order in evaluating $\underline{P}^{DF}(\underline{H})$ is due to the following reasons. From Lemma 1, for any polymatroids defined by the set functions f_1 and f_2, an inactive case results when

$$f_1(\mathcal{S}) + f_2(\mathcal{K}\backslash\mathcal{S}) < \min(f_1(\mathcal{K}), f_2(\mathcal{K})) \quad \text{for a } \mathcal{S} \subset \mathcal{K}. \tag{51}$$

Thus, the condition in (51) for the inactive cases by definition precludes an active case. For $K = 2$, these conditions simplify to those in (40) and (41) for cases 1 and 2, respectively. Furthermore, the inactive cases are also mutually exclusive. The remaining (active and boundary) cases satisfy the conditions

$$\text{Cases } 3a, 3b, 3c: \quad f_1(\mathcal{S}) + f_2(\mathcal{K}\backslash\mathcal{S}) > \min(f_1(\mathcal{K}), f_2(\mathcal{K})) \tag{52}$$

$$\text{Boundary Cases}:$$

$$f_1(\mathcal{S}) + f_2(\mathcal{K}\backslash\mathcal{S}) = \min(f_1(\mathcal{K}), \quad f_2(\mathcal{K})) \quad \text{for one } \mathcal{S}^* \subset \mathcal{K} \tag{53}$$

$$f_1(\mathcal{S}) + f_2(\mathcal{K}\backslash\mathcal{S}) > \min(f_1(\mathcal{K}), f_2(\mathcal{K})) \quad \text{for all } \mathcal{S}(\neq \mathcal{S}^*) \subset \mathcal{K}.$$

For $K = 2$, the condition in (52) simplifies to those in (42)–(44) for cases $3a$, $3b$, and $3c$, respectively, while that in (53) simplifies to those in (45)–(50) for the respective boundary cases. Additionally, for cases $3a, 3b$, and $3c$, we also have the requirement that the sum-rate at the relay is less than, greater than, and equal to that at the destination, respectively. The conditions in (51) and (52) are mutually exclusive. On the other hand, the equality condition for a boundary case (l, n), for all l, n, is subsumed in the optimization while the inequality condition is satisfied for all except one subset of users $\mathcal{S}^*$ for which the equality condition holds. This in turn implies that case (l, n) has one less inequality condition than case n. Since case $3c$ has no inequality conditions, neither do cases $(1, 3c)$ and $(2, 3c)$. Thus, the optimality of cases $(1, 3c)$ and $(2, 3c)$ can be determined only after eliminating the optimality of all others just as the optimality of case $3c$ is determined after that of cases $3a$ and $3b$. The order of all other active and boundary cases can be chosen arbitrarily, and for ease of presentation, we simply assume that the search algorithm first verifies the optimality of $\underline{P}^{(1)}(\underline{H})$, failing which it verifies the optimality of $\underline{P}^{(2)}(\underline{H})$, followed by $\underline{P}^{(3a)}(\underline{H})$, $\underline{P}^{(3b)}(\underline{H})$, and $\underline{P}^{(3c)}(\underline{H})$, and finally verifies the optimality of the boundary cases in the order $(1, 3a)$, $(2, 3a)$, $(1, 3b)$, $(2, 3b)$, $(1, 3c)$, and $(2, 3c)$. Note, however, that cases $(1, 3c)$ and $(2, 3c)$ are mutually exclusive due to cases 1 and 2 being disjoint. Thus, the optimal $\underline{P}^*(\underline{H})$ is only achieved by a unique $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$ depending on the policy that satisfies its case conditions. ∎

Remark 6: The conditions for cases $3a$, $3b$, and $3c$ can also be redefined to include the negation of all the conditions for the other cases. This in turn eliminates the need for an order in computing the optimal policy; however, the number of conditions that need to be checked to verify whether the optimal policy satisfies the conditions for cases $3a$ or $3b$ or $3c$ remain unchanged relative to the algorithm in Theorem 2.

We now summarize the optimal power policies at the sources and the relay for the different cases as follows.

Optimal Relay Policy: In the orthogonal model we consider, the relay transmits directly to the destination on a channel orthogonal to the source transmissions. Thus, the relay to destination link can be viewed as a fading point-to-point link. In fact, in all cases the optimal relay policy involves water-filling over the fading states analogous to a fading point to point link (see [27]). However, the exact solution, including scale factors, depends on the case considered. Thus, for case 1, maximizing the sum rate results in the relay using its power $P_r(\underline{H})$ to forward only the message from user 1 in every fading state in which it transmits. Similarly, for case 2, the relay cooperates entirely with user 2. For the active cases, $3a$ and $3b$, the sum-rate may be achieved by an infinite number of feasible points on one or both of the sum-rate planes; the optimal cooperative strategy at the relay will differ for each such point. Thus, for a corner point the relay transmits a message from only one of the users while for all noncorner points the relay transmits both messages. For the boundary cases, the water-filling solution at the relay is dependent on the Lagrangian parameter(s) introduced to satisfy the boundary conditions.

Optimal User Policies: As with the relay, the optimal policies for the two users depend on the case considered. For cases 1 and 2, the optimal policies are water-filling solutions to that receiver at which it achieves a lower rate. In fact, the conditions for case 1 in (40) suggest a network geometry in which source 1 and the relay are physically proximal enough to form a *cluster* and source 2 and the destination form another cluster; and vice-versa for case 2. For cases $3a$ and $3b$, the optimal policies at the two users maximize the two-user multiple-access sum-rate (see [25], [26]) achieved at the relay and destination, respectively, and thus, the optimal policy for each user involves water-filling over its fading states to that receiver. The solution also exploits the multi-user diversity to opportunistically schedule the users in each use of the channel. The optimal policies for case $3c$ require the users to allocate power such that the sum-rates achieved at both the relay and the destination are the same. This constraint has the effect that it preserves the opportunistic scheduling since the sum-rate involves the multi-access sum-rate bounds at both receivers. However, the solutions are no longer water-filling due to the fact that the equality (boundary) condition results in the function $f_k^{(3c)}$ being a weighted sum of the functions $f_k^{(3a)}$ and $f_k^{(3b)}$ for cases $3a$ and $3b$, respectively. The same observation holds true for the boundary cases too since $f_k^{(l,n)}$ are weighted sums of the functions for cases l and n.

Remark 7: The case conditions in (40)–(50) require averaging over the channel states; thus, the case that maximizes the sum-rate depends on the average power constraints and the channel statistics (including network topology).

Remark 8: The optimal policy for each source for cases 1, 2, $3a$, and $3b$ depends on the channel gains at only one of the receivers. However, the optimal policy for the boundary cases, including case $3c$, depends on the instantaneous channel states at both receivers. Furthermore, all the cases exploiting the multiuser diversity require a centralized protocol to coordinate the opportunistic scheduling of users.

V. K-User Generalization and DF Rate Region

A. K-User Sum-Rate Analysis

We use Lemma 1 to extend the two-user analysis in Section IV to K users (and a fixed θ). Recall that $\mathcal{R}_{DF}^{\theta}$ in Theorem 1 is given by a union of the intersection of polymatroids, where the union is over all power policies. From Lemma 1, we have that the maximal K-user sum-rate tuple is achieved by an intersection that either belongs to active set or to the inactive set. We index the 2^K-2 nonempty subsets $\mathcal{S}^{(l)}$ of $\mathcal{K}$ via a superscript $l=1,2,\ldots,2^K-2$. For a K-user MARC, there are (2^K-2) possible intersections of the inactive kind with sum-rate $S^{(l)}$ given by

$$\underline{\text{case } l}:\ S^{(l)} = R_{\mathcal{S}^{(l)},r} + R_{\mathcal{K}\backslash\mathcal{S}^{(l)},d},\ \text{s.t. } R_{\mathcal{S}^{(l)},r} < R^{\min}_{\mathcal{S}^{(l)},d} \text{ and } R_{\mathcal{K}\backslash\mathcal{S}^{(l)},d} < R^{\min}_{\mathcal{K}\backslash\mathcal{S}^{(l)},d},\quad l=1,2,\ldots,2^K-2 \tag{54}$$

where $R_{\mathcal{A},j}$ and $R^{\min}_{\mathcal{A},j}$ are as defined in Section III and for $j=r,d$, are given by the bounds (11) and (12), respectively. The sum-rates $S^{(i)}$, for the active cases $i=3a,3b,3c$, are

$$\underline{\text{case } 3a}:\ S^{(3a)} = R_{\mathcal{K},r}\quad \text{s.t. } R_{\mathcal{K},r} < R_{\mathcal{K},d} \text{ and } R_{\mathcal{K},r} < S^{(l)} \text{ for all } l \tag{55}$$

$$\underline{\text{case } 3b}:\ S^{(3b)} = R_{\mathcal{K},d}\quad \text{s.t. } R_{\mathcal{K},d} < R_{\mathcal{K},r} \text{ and } R_{\mathcal{K},d} < S^{(l)} \text{ for all } l \tag{56}$$

$$\underline{\text{case } 3c}:\ S^{(3c)} = R_{\mathcal{K},r}\quad \text{s.t. } R_{\mathcal{K},d} = R_{\mathcal{K},r} < S^{(l)} \text{ for all } l. \tag{57}$$

Finally, the sum-rate $S^{(l,n)}$, for the boundary cases totaling $3(2^K-2)$ and enumerated as cases (l,n), $l=1,2,\ldots,2^K-2$, $n=3a,3b,3c$, are

$$\underline{\text{case } (l,3a)}:\ S^{(l,3a)} = R_{\mathcal{K},r}$$
$$\text{s.t. } R_{\mathcal{K},r} = R_{\mathcal{S}^{(l)},r} + R_{\mathcal{K}\backslash\mathcal{S}^{(l)},d} = S^{(l)} < R_{\mathcal{K},d} \tag{58}$$
$$\underline{\text{case}(l,3b)}:\ S^{(l,3b)} = R_{\mathcal{K},d}$$
$$\text{s.t. } R_{\mathcal{K},d} = R_{\mathcal{S}^{(l)},r} + R_{\mathcal{K}\backslash\mathcal{S}^{(l)},d} = S^{(l)} < R_{\mathcal{K},r} \tag{59}$$
$$\underline{\text{case}(l,3c)}:\ S^{(l,3c)} = R_{\mathcal{K},r}$$
$$\text{s.t. } R_{\mathcal{K},r} = R_{\mathcal{K},d} = R_{\mathcal{S}^{(l)},r} + R_{\mathcal{K}\backslash\mathcal{S}^{(l)},d} = S^{(l)} \tag{60}$$

where the subset $\mathcal{S}$ is chosen to correspond to the appropriate case l.

Remark 9: The constraint for case l in (54) are obtained directly from the requirement that the K-user sum-rate constraints at the two receivers are larger than that for case l (see (51)).

The K-user sum-rate optimization problem for cases i and (l,n) can be written as

$$\boxed{\begin{array}{c}\max_{\underline{P}(\underline{H})\in\mathcal{P}} S^{(i)} \text{ or } \max_{\underline{P}(\underline{H})\in\mathcal{P}} S^{(l,n)} \\ \text{s.t. } \ \mathbb{E}[P_k(\underline{H})] \le \bar{P}_k, \\ P_k(\underline{h}) \ge 0,\ k=1,2,\ldots,K,r, \text{ and } for\ all\ \underline{h}.\end{array}} \tag{61}$$

An inactive case l results when the conditions for that case in (54) are satisfied. The active cases $3a$, $3b$, and $3c$ result when the conditions in (55), (56), and (57) are satisfied, respectively. A boundary case results when the K-user sum-rate for case n is equal to that for case l as indicated in (58)–(60). Finally, as before, the achievable maximum sum-rate is given by (15) and (16) when θ can be varied. The DF sum-rate optimization problem here is analagous to the two-user case and in the interest of space, we simply summarize our results in the following theorem.

Theorem 3: The optimal power policy $\underline{P}^*(\underline{H})$ that maximizes the DF sum-rate of a K-user ergodic fading orthogonal Gaussian MARC is obtained by computing $\underline{P}^{(i)}(\underline{H})$ and $\underline{P}^{(l,n)}(\underline{H})$ starting with the inactive cases $1,2,\ldots,2^K-2$, followed by the active cases $3a, 3b$, and $3c$, and finally the boundary cases (l,n), choosing cases $(l,3c)$ after computing $\underline{P}^{(l,n)}(\underline{H})$ for cases $(l,3a)$ and $(l,3b)$ for all l, until for some case the corresponding $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$ satisfies the case conditions. The optimal $\underline{P}^{DF}(\underline{H})$ is given by the optimal $\underline{P}^{(i)}(\underline{H})$ or $\underline{P}^{(l,n)}(\underline{H})$ that satisfies its case conditions and falls into one of the following three categories:

Inactive Cases: The optimal user policy $P_k^{DF}(\underline{H})$, for all $k\in\mathcal{K}$, is multi-user opportunistic water-filling over its bottle-neck (rate limiting) link to the relay among users in $\mathcal{S}$ or the destination among users in $\mathcal{K}\backslash\mathcal{S}$. The optimal relay policy $P_r^{DF}(\underline{H})$ is water-filling over its direct link to the destination.

Active Cases $(3a, 3b, 3c)$*:* The optimal user policy $P_k^{DF}(\underline{H})$, for all $k\in\mathcal{K}$, is opportunistic water-filling over its link to the relay for case $3a$ and to the destination for case $3b$. For case $3c$, $P_k^{DF}(\underline{H})$, for all $k\in\mathcal{K}$, takes an opportunistic nonwater-filling form. The optimal relay policy P_r^{DF} is water-filling over the relay-destination link.

Boundary Cases: The optimal user policy $P_k^{DF}(\underline{H})$, for all $k \in \mathcal{K}$, takes an opportunistic nonwater-filling form. The optimal relay policy $P_r^{DF}(\underline{H})$ is water-filling over its direct link to the destination.

Based on the optimal DF policies, one can conclude that the topology of the network affects the form of the solution with the classic multi-user opportunistic water-filling solutions applicable only for the sources-relay or the relay-destination clustered models. For all other partially clustered or nonclustered networks, the solutions are a combination of single- and multi-user water-filling and nonwater-filling but opportunistic solutions.

B. K-User Rate Region

Analogously to the two-user analysis, one can also generalize the sum-rate analysis above to derive the optimal policies for all points on the boundary of the K-user DF rate region. For brevity, we outline the approach below.

We start with the observation that the DF rate region, $\mathcal{R}_{DF}$, is convex, and thus, every point on the boundary of $\mathcal{R}_{DF}$ is obtained by maximizing the weighted sum $\sum_{k\in\mathcal{K}}\mu_k R_k$, $\mu_k > 0$ for all k. As noted earlier, each point on the boundary of $\mathcal{R}_{DF}$ is obtained by an intersection of two polymatroids for some $\underline{P}(\underline{H})$. Thus, analogously to the sum-rate analysis for $\mu_k = 1$ for all k, for arbitrary $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_K)$, $\sum_{k\in\mathcal{K}}\mu_k R_k$, is maximized by either an inactive or an active case.

Since the maximum value of $\sum_{k\in\mathcal{K}}\mu_k R_k$ over a feasible bounded polyhedron is achieved at a vertex of the polyhedron, for any $\underline{P}(\underline{H})$, the $(R_1, R_2, \ldots, R_K)$-tuple maximizing $\sum_{k\in\mathcal{K}}\mu_k R_k$ is given by a vertex of an $\mathcal{R}_r(\underline{P}(\underline{H}))\cap\mathcal{R}_d(\underline{P}(\underline{H}))$ polyhedron at which $\sum_{k\in\mathcal{K}}\mu_k R_k$ is a tangent. For the $2^K - 2$ inactive cases, the polymatroid intersections are polytopes with constraints on the multi-access rates of all users in $\mathcal{S}$ and $\mathcal{K}\backslash\mathcal{S}$ at the relay and destination, respectively. Since bounds on the multi-access rates of l users result in a polymatroid with $l!$ vertices, the intersection of the two orthogonal sum-rate planes will result in a polytope with $(|\mathcal{S}|!)(|\mathcal{K}\backslash\mathcal{S}|!)$ vertices of which an appropriate vertex will maximize $\sum_{k\in\mathcal{K}}\mu_k R_k$. Each of the $3(2^K - 2)$ boundary cases are also characterized by an intersection with $(|\mathcal{S}|!)(|\mathcal{K}\backslash\mathcal{S}|!)$ vertices since these active cases are such that only one point on the sum-rate plane is included in the region of intersection. Finally, for cases $3a$, $3b$, and $3c$, the intersection of K-dimensional polymatroids results in a K-dimensional polyhedron.

In general, the intersection of two polymatroids is not a polymatroid, and thus, unlike the case with polymatroids, greedy algorithms do not maximize the weighted sum of rates. This in turn implies that closed form expressions are not in general possible and determining the optimal power policies requires convex programming techniques. We comment specifically on two cases of most interest.

Remark 10: For the special case in which the optimal policies for all $\underline{\mu}$ are such that the bounds at the relay are smaller than the bounds at the destination for all $\mathcal{S}$, i.e., $\mathcal{R}_r^\theta \subset \mathcal{R}_d^\theta$, the optimal user policies for all $\underline{\mu}$ are multi-user water-filling solutions developed in [26, II.C] with the relay as the receiver. Note that this condition implies that all possible subsets of users achieve better rates at the destination than at the relay. This can happen when either all users are clustered closer to the destination or when the relay has a relatively high SNR link to the destination sufficient enough to achieve rate gains for all users at the destination. This case is interesting only if the rates achieved thus are larger than the MAC sum-capacity (without relay).

Remark 11: Similarly, for the special case in which the optimal policies for all $\underline{\mu}$ are such that $\mathcal{R}_d^\theta \subset \mathcal{R}_r^\theta$, the optimal user policies are multi-user water-filling solutions with the destination as the receiver. This case occurs when case $3b$ holds for all points on the boundary of the DF rate region. This condition implies that all possible subsets of users achieve better rates at the relay than they do at the destination which in turn suggests a geometry in which all subsets of users are clustered closer to the relay than to the destination. The optimal relay policy in all cases is a water-filling solution over its link to the destination. In the following section we show that for this case DF achieves the capacity region.

VI. Outer Bounds

Thus far, we have focused on the DF achievable scheme. It is worthwhile to understand the conditions under which DF can achieve the sum-capacity, and if possible, the capacity region, for an ergodic fading Gaussian MARC. To this end, we develop outer bounds for this channel using cut-set bounds. Specifically, we obtain our outer bounds by specializing the known cut-set bounds developed in [10] for a K-user half-duplex discrete memoryless (d.m.) MARC to the Gaussian case. We summarize these half-duplex d.m. bounds summarized below. As with DF, we focus on the case in which the bandwidth parameter θ is fixed *a priori*, and thus, is not part of the optimization of the outer bound rate region. For the case in which θ can be varied, the rate region will be a union over regions, one for each feasible θ.

Proposition 3: For the orthogonal MARC with a fixed θ the capacity region is contained in the union of the set of rate tuples $(R_1, R_2, \ldots, R_K)$ that satisfy

$$R_{\mathcal{S}} \leq \min\{\theta I(X_{\mathcal{S}}; Y_r Y_{d,1} \,|\, X_{\mathcal{S}^c}, U), \theta I(X_{\mathcal{S}}; Y_{d,1} \,|\, X_{\mathcal{S}^c}, U) + \bar{\theta} I(X_r; Y_{d,2}|U)\}, \quad \text{for all } \mathcal{S} \subseteq \mathcal{K} \quad (62)$$

where the union is taken over all distributions that factor as

$$p(u)\left(\prod_{k=1}^{K} p(x_k \,|\, u)\right) p(x_r \,|\, u) p(y_r y_{d,1} \,|\, x_{\mathcal{K}}) p(y_{d,2} \,|\, x_r). \quad (63)$$

Remark 12: The *time-sharing* random variable $U \in \mathcal{U}$ ensures that the region defined by (62) is convex. One can apply Caratheodory's theorem [38] to this K-dimensional convex region to bound the cardinality of $\mathcal{U}$ as $|\mathcal{U}| \leq K + 1$.

Following techniques similar to those for proving the converse for Gaussian MAC, we obtain

$$R_{\mathcal{S}} \leq \min \left\{ \theta \log \left| \mathbf{\Sigma} + \sum_{k \in \mathcal{S}} \mathbf{g}_k P_k/\theta \right| \right.$$
$$\left. \theta C \left(\sum_{k \in \mathcal{S}} |h_{d,k}|^2 P_k/\theta \right) + \bar{\theta} C(|h_{d,r}|^2 P_r/\bar{\theta}) \right\} \quad (64)$$

where Σ is the covariance matrix of a noise vector $Z = [Z_r \; Z_{d,1}]^T$

$$\mathbf{g}_k = [\, h_{r,k} \quad h_{d,k} \,]^T [\, h_{r,k}^* \quad h_{d,k}^* \,] \quad (65)$$

$h_{(\cdot)}^*$ is the complex conjugate of $h_{(\cdot)}$, and a conditional entropy theorem [39] is used to show that Gaussian signals $X_k \sim \mathcal{CN}(0, P_k/\theta)$, $k = 1, 2$, and $X_r \sim \mathcal{CN}(0, P_r/\bar{\theta})$ maximize the bounds in (62). Using the fact that the ergodic channel is a collection of parallel nonfading channels, one for each fading state instantiation, the capacity region of an ergodic fading orthogonal Gaussian MARC is as described in the following theorem.

Theorem 4: The capacity region $\mathcal{C}_{O-\mathrm{MARC}}^{\theta}$ of an ergodic fading orthogonal Gaussian MARC with a fixed bandwidth parameter θ is contained in

$$\mathcal{R}_{OB}^{\theta}(\underline{\bar{P}}) = \bigcup_{\underline{P}(\underline{H}) \in \mathcal{P}} \left\{ \mathcal{R}_1^{\theta}(\underline{P}(\underline{H})) \cap \mathcal{R}_2^{\theta}(\underline{P}(\underline{H})) \right\} \quad (66)$$

where, for all $\mathcal{S} \subseteq \mathcal{K}$, we have

$$\mathcal{R}_1^{\theta}(\underline{P}(\underline{H})) = \left\{ (R_1, R_2) : R_{\mathcal{S}} \leq \mathbb{E} \right.$$
$$\left. \times \left[\theta \log \left| I + \sum_{k \in \mathcal{S}} \mathbf{G}_k P_k(\underline{H}) \middle/ \theta \right| \right] \right\} \quad (67)$$

where

$$\mathbf{G}_k = [\, H_{r,k} \quad H_{d,k} \,]^T [\, H_{r,k}^* \quad H_{d,k}^* \,] \quad (68)$$

and

$$\mathcal{R}_2^{\theta}(\underline{P}(\underline{H})) = \left\{ (R_1, R_2) : R_{\mathcal{S}} \right.$$
$$\leq \mathbb{E} \left[\theta C \left(\sum_{k \in \mathcal{S}} |H_{d,k}|^2 P_k(\underline{H})/\theta \right) \right.$$
$$\left. \left. + \bar{\theta} C \left(|H_{d,r}|^2 \, P_r(\underline{H})/\bar{\theta} \right) \right] \right\}. \quad (69)$$

Remark 13: Comparing outer bounds in (69) with the DF bounds in (12), we see that the bounds at the destination are the same in both cases. However, unlike the DF bound at only the relay in (11), the cutset bound in (67) is a single-input multiple-output (SIMO) bound with single-antenna transmitters and with the relay and the destination acting as a multiantenna receiver.

The expressions in (67) and (69) are concave functions of $P_k(\underline{H})$, for all k, and thus, the region $\mathcal{R}_{OB}$ is convex. Thus, as in Theorem 1, the region $\mathcal{R}_{OB}$ in (66) is a union of the intersections of the regions $\mathcal{R}_1(\underline{P}(\underline{H}))$ and $\mathcal{R}_2(\underline{P}(\underline{H}))$, where the union is taken over all $\underline{P}(\underline{H}) \in \mathcal{P}$ and each point on the boundary of $\mathcal{R}_{DF}$ is obtained by maximizing the weighted sum $\mu_1 R_1 + \mu_2 R_2$ over all $\underline{P}(\underline{H}) \in \mathcal{P}$, and for all $\mu_1 > 0$, $\mu_2 > 0$. In [40], it is shown that the rate polytopes satisfying the full-duplex cutset bounds are polymatroids. Since the polytopes in (67) and (69) are obtained from the full-duplex case for the special case of orthogonal signaling, one can verify in a straightforward manner using Definition 1 that these are polymatroids as well.

A. Optimal Sum-Rate Policies and Sum-Capacity

Since $\mathcal{R}_{OB}^{\theta}$ is obtained completely as a union of the intersection of polymatroids, one for each choice of power policy, Lemma 1 can be applied to explicitly characterize the outer bounds on the sum-rate. Thus, the maximum sum-rate tuple is achieved by an intersection that belongs to either the active set or to the inactive set such that there are $2^K - 2$ inactive cases, cases $3a$, $3b$, and $3c$, and $3(2^K - 2)$ boundary cases. The analysis here is analagous to the K-user DF case and the optimization for each case involves writing the Lagrangian and the KKT conditions. The optimal policy $\underline{P}^{(ob)}(\underline{H})$ satisfies the conditions for only one of the cases. Comparing these optimal policies with that for DF, we have the following capacity theorem.

Theorem 5: The sum-capacity of a K-user ergodic fading orthogonal Gaussian MARC is achieved by DF when the optimal policies $\underline{P}^{(ob)}(\underline{H})$ and $\underline{P}^{DF}(\underline{H})$ for the cutset and DF bounds, respectively, satisfy the conditions for case $3b$ and for no other case.

Proof: The proof follows from comparing the sum-rate expressions for all cases for the inner and outer bounds, respectively. For all those cases in which the SIMO cut-set bound dominates the sum-rate, the cutset bounds do not match the DF bounds. On the other hand, when the optimal policies $\underline{P}^{(ob)}(\underline{H})$ and $\underline{P}^{DF}(\underline{H})$ satisfy the conditions for case $3b$, the bound on $R_{\mathcal{K}}$ at the destination dominates for both the inner and the outer bounds. Furthermore, since this sum-rate bound at the destination is the same for both DF and the outer bounds, we have $\underline{P}^{(ob)}(\underline{H}) = \underline{P}^{DF}(\underline{H}) = \underline{P}^{(3b)}(\underline{H})$, and thus, DF achieves the sum-capacity. ■

Remark 14: Recall that case $3b$ corresponds to a clustered geometry in which the relay is clustered with all sources such that the cooperative multi-access link from the sources and the relay to the destination is the bottleneck link.

Remark 15: The set of power policies, $\mathcal{B}^{(i)}$ and $\mathcal{B}^{(l,n)}$, are defined by the appropriate conditions for the DF and outer bounds which are not necessarily the same (since the bounds are not exactly the same). However, when case $3b$ maximizes both the inner and outer bounds, we have $\underline{P}^{DF}(\underline{H}) = \underline{P}^{(ob)}(\underline{H}) = \underline{P}^{(3b)}(\underline{H}) \in \mathcal{B}^{(3b)}$ for both bounds.

B. Capacity Region

One can similarly write the rate expressions and the KKT conditions for every point on the boundary of $\mathcal{R}_{OB}^{\theta}$. The analysis is similar to that for the K-user orthogonal MARC under DF developed in Section V-B. From Theorem 4, every point

$\sum_{k\in\mathcal{K}}\mu_k R_k$ on $\mathcal{R}_{OB}^{\theta}$ results from an intersection of two polymatroids. For those cases in which the intersection is an inactive case, both the SIMO cut-set bound at the relay and destination and the cooperative cut-set bound at the destination are involved, and thus, one cannot achieve capacity. This is also true for the boundary cases. For cases $3a$, $3b$, and $3c$, in which the polymatroid intersection also has 2^K-1 constraints, and hence, $K!$ corner points on the dominant K-user sum-rate face, $\sum_{k\in\mathcal{K}}\mu_k R_k$ is maximized by a corner point (vertex) of the resulting polytope. Any polytope that results from some or all of the SIMO bounds will be larger than the corresponding DF inner bounds. On the other hand, since $\mathcal{R}_d^{\theta}$ and $\mathcal{R}_2^{\theta}$, the DF and outer bound rate regions, respectively, at the destination are the same, the cut-set bounds can be tight when these bounds dominate the weighted rate sum. This occurs when the power policies maximizing the DF and outer outer bounds result in case $3b$. Thus, for the cut-set bounds to be tight for a given $\underline{\mu}$ and θ, we require

$$\mathcal{R}_2^{\theta}\left(\underline{P}^{(ob)}(\underline{H},\underline{\mu},\theta)\right)\subset\mathcal{R}_1^{\theta}\left(\underline{P}^{(ob)}(\underline{H},\underline{\mu},\theta)\right),\quad\text{and}\tag{70a}$$

$$\mathcal{R}_d^{\theta}\left(\underline{P}^{DF}(\underline{H},\underline{\mu},\theta)\right)\subset\mathcal{R}_r^{\theta}\left(\underline{P}^{DF}(\underline{H},\underline{\mu},\theta)\right)\tag{70b}$$

where $\underline{P}^{(ob)}(\cdot)$ and $\underline{P}^{DF}(\cdot)$ denote the power policies maximizing $\sum_{k\in\mathcal{K}}\mu_k R_k$ for DF and the outer bounds, respectively, for a given θ. Furthermore, when (70) is satisfied, we have $\underline{P}^{(ob)}(\cdot)=\underline{P}^{DF}(\cdot)=\underline{P}^{(3b)}(\cdot)$, i.e., DF achieves the optimal weighted sum of rates $\sum_{k\in\mathcal{K}}\mu_k R_k$ for a given $\underline{\mu}$. When this requirement holds for all $\underline{\mu}$, we obtain the capacity region. This observation is summarized in the following theorem.

Theorem 6: The capacity region $\mathcal{C}_{O-\text{MARC}}^{\theta}$ of an ergodic orthogonal Gaussian MARC with a fixed θ is achieved by DF when every point $\sum_{k\in\mathcal{K}}\mu_k R_k$ in the DF and the outer bound rate regions is achieved by case $3b$ such that $\underline{P}^{(ob)}(\underline{H},\mu_1,\mu_2,\theta)=\underline{P}^{DF}(\underline{H},\mu_1,\mu_2,\theta)\equiv\underline{P}^{*}(\underline{H},\mu_1,\mu_2,\theta)$. The requirement that case $3b$ maximizes every point on the boundary of both the DF and outer bound rate regions implies that for both bounds, $\mathcal{R}_d^{\theta}(\underline{P}^*(\cdot))\subset\mathcal{R}_r^{\theta}(\underline{P}^*(\cdot))$ and $\mathcal{R}_2^{\theta}(\underline{P}^*(\cdot))\subset\mathcal{R}_1^{\theta}(\underline{P}^*(\cdot))$ such that

$$\mathcal{C}_{O-\text{MARC}}^{\theta}=\mathcal{R}_d^{\theta}(\underline{P}^*(\cdot))=\mathcal{R}_2^{\theta}(\underline{P}^*(\cdot)).\tag{71}$$

Remark 16: For the special case in which $\underline{H}$ has uniform phase fading and the channel state information is not known at the transmitters such that $P_k^{(ob)}(\underline{H})=P_k^{(3b)}(\underline{H})=\bar{P}_k$, for all $k\in\mathcal{T}$, Theorem 6 yields the capacity region of an ergodic phase fading orthogonal Gaussian MARC as developed in [3, Theorem 9].

Remark 17: For the case of variable θ, the capacity region is still achieved only when case $3b$ maximizes all possible weighted sum of rates for both the DF and the outer bounds. However, now the rate regions result from a maximization over all feasible θ. For the case of choosing the same sum-rate maximizing θ^* over all channel states, this optimization involves searching over the entire range of $\theta\in[0,1]$, because the equation that the optimal θ satisfies involves averaging rates over channel states, and has to be computed numerically. However, for the case with θ varying with channel state, for each channel state, obtaining θ^* can be done more easily by solving an equation for the root instead of an exhaustive search (see, for e.g., [6] for $K=1$). For time-duplexed relay systems in [21], a quasi-concavity property of the sum-rate as a function of the time-sharing fraction is exploited to determine the sum-rate maximizing time fraction.

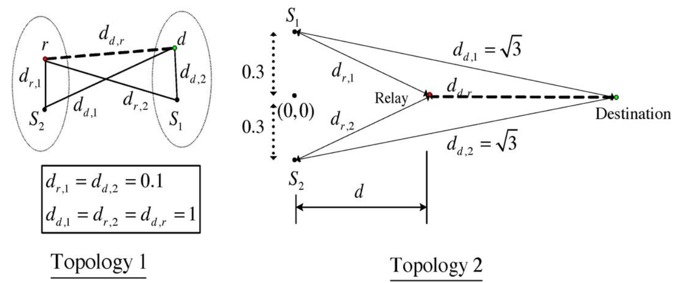

Fig. 6. Two-user MARCs with partially clustered (topology 1) and symmetric (topology 2) topologies.

VII. Illustration of Results

We present numerical results for a two-user orthogonal MARC with Rayleigh fading links. We model the channel fading gains between receiver m and transmitter k, for all k and m, as

$$H_{m,k}=\frac{A_{m,k}}{\sqrt{d_{m,k}^{\gamma}}}\tag{72}$$

where $d_{m,k}$ is the distance between the transmitter and receiver, γ is the path-loss exponent, and $A_{m,k}$ is a circularly symmetric complex Gaussian random variable with zero mean and unit variance such that $|H_{m,k}|^2$ is Rayleigh distributed with zero mean and variance $1/d_{m,k}^{\gamma}$. We assume that $A_{m,k}$ are independent for all m,k. For the purpose of our illustration, we set the path-loss exponent $\gamma=3$. Finally, we use Monte Carlo methods to simulate the random parameters and evaluate the sum-rates for the inner DF and outer bounds.

Towards illustrating our results, we consider a two-user MARC under two different topologies as shown in Fig. 6. The first topology models a partially clustered MARC in which one of the users (user 1) is closer to the relay while the other (user 2) is closer to the destination than the other receiver. The second topology models a symmetric geometry where the users are equidistant from the destination and the relay. the average power levels in dB for topology 1 and 2 are $\bar{P}_1=\bar{P}_2=0$ dB, $\bar{P}_r=6$ dB and $\bar{P}_1=\bar{P}_2=\bar{P}_r=4$ dB, respectively.

The partially clustered topology, referred to as topology 1, in Fig. 6 models Case 1 in our analysis. For this topology, we plot the rate regions achieved at the relay and destination for three values of $\theta=0.25, 0.5$, and 0.75 in Fig. 7. Also shown in each sub-plot (one for each θ) is the MAC capacity region without resource allocation, i.e., assuming transmission at the same average power in each use of the channel. For all three choices of θ, the optimal policies are those satisfying the conditions for case 1 such that users 1 and 2 waterfill over their bottleneck links to the destination and relay, respectively. However, as shown in Fig. 7, for all these choices of θ, the DF sum-rate is strictly smaller than

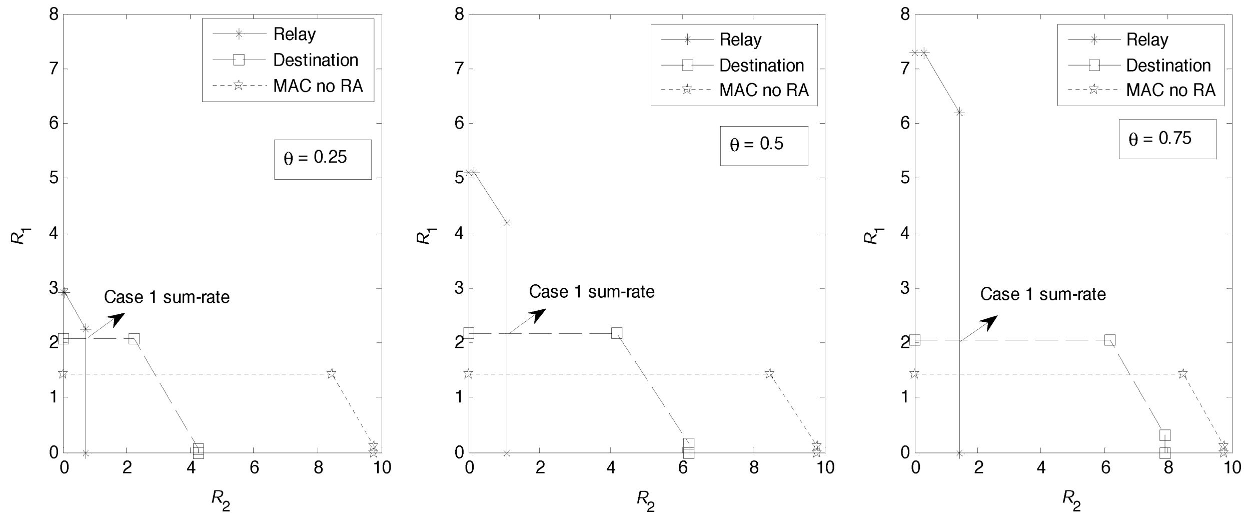

Fig. 7. Plot of the DF rate regions at the relay and destination and that of the MAC without resource allocation for $\theta = 0.25$, 0.5, and 0.8.

the sum-rate of the ergodic fading Gaussian MAC (from sources to the destination) without resource allocation. This is due to two limitations: a) both users in the MARC are waterfilling to the distant receivers while in the MAC user 2 has the advantage of being closer to the intended receiver, and b) while the relay aids user 1, the fractional bandwidth use resulting from the orthogonal half-duplex model significantly limits the rate gains. We make further comments on strategies for such geometries following the discussion of the second topology below.

Consider Topology 2 in Fig. 6. For the symmetric geometry modeled, in Fig. 8 we plot the inner (DF) and outer cutset bounds on the sum-rate for $\theta = 1/2$ as a function of the relay position along the horizontal axis. As a result of the symmetric geometry, for every choice of the relay position, both the inner and outer bounds on the sum-rate are maximized by one of cases $3a$, $3b$, or $3c$. For each case, we use an iterative algorithm, as described in the Appendix, to compute the sum-rate maximizing user policies. For cases $3a$ and $3b$, the iterative algorithm simplifies to the iterative water-filling algorithm developed in [41] in which at each step the algorithm finds the single-user water-filling policy for each user while regarding the signals from the other user as noise. For case $3c$, the optimal policy at each step is still obtained by regarding the signals from the other user as noise; however, the user policy at each step is no longer a water-filling solution. Finally, the optimality of DF when the sources are clustered relatively closer to the relay than to the destination is amply demonstrated in Fig. 8. The inner and outer bounds are also compared with the sum-capacity of the fading multi-access channel without a relay and $\theta = 1$, shown by the dashed line that is a constant independent of the relay position. Also shown in Fig. 8 are the ranges of relay positions for cases $3a$, $3b$, and $3c$ for both DF and the cutset bounds.

For case $3a$ in which the multiple access channel to the relay from the sources is the bottleneck link, for relay positions very close to the destination (>0.8 in Fig. 8), the DF sum rate achieved for $\theta = 1/2$ is strictly smaller than the waterfilling sum-capacity of the multiple access channel from the users to the destination (without the relay). A similar behavior is seen for the outer bound sum-rate when the relay is very proximal to the destination. This limitation is due to the orthogonal half-duplex constraint on the sources. One approach to increase the rate is to increase θ (as discussed below). The sum rate can also be potentially increased by allowing the sources to transmit in both orthogonal bands; however, it comes at a cost of increased complexity of analysis and solutions. In general, however, as has been established in earlier works (e.g., [3], [42]), a decode-and-forward relay provides significant rate and capacity benefits when it is physically closer to all the sources.

Finally, in Fig. 9 we illustrate the effect of three values of $\theta = 0.4, 0.5$, and 0.75 for Topology 2. Relative to $\theta = 1/2$, for $\theta > 1/2$, the cooperative multiple-access link from the sources and relay to the destination (case $3b$) remains a bottleneck link for a larger range of relay positions thereby increasing the range over which the sum capacity will be achieved. This is because for larger θ, the rate achieved at the relay is larger. However, the smaller bandwidth allocated to the relay to destination link results in smaller case $3b$ sum rates relative to the $\theta = 0.4$ and the 0.5 curves. On the other hand, for relay positions closer to the destination, where the multiple access sum rate from the sources to the relay is the bottleneck link, the rates achieved are larger relative to the other two plots since θ is larger. The observations hold in reverse for $\theta = 0.4$ relative to the larger values of θ. The different choices of θ suggest that θ could be chosen a priori to maximize the rates achieved for a given network geometry. In all cases, as the relay approaches the destination, the performance of DF for this orthogonal signaling scheme we consider falls below that of the MAC sum-capacity.

Observations: The results illustrated thus far lead us to make the following observations:

1) The relay is most useful to all users only when all the users are clustered close to the relay such that the combined channel from the users and the relay to the destination is the bottleneck link.
2) Partially clustered geometries suffer from the limitation that users that are more distant from the relay than their intended receiver also need to be decoded by the relay. This in addition to the orthogonal half-duplex constraint limits the rates achieved significantly. One mitigating approach

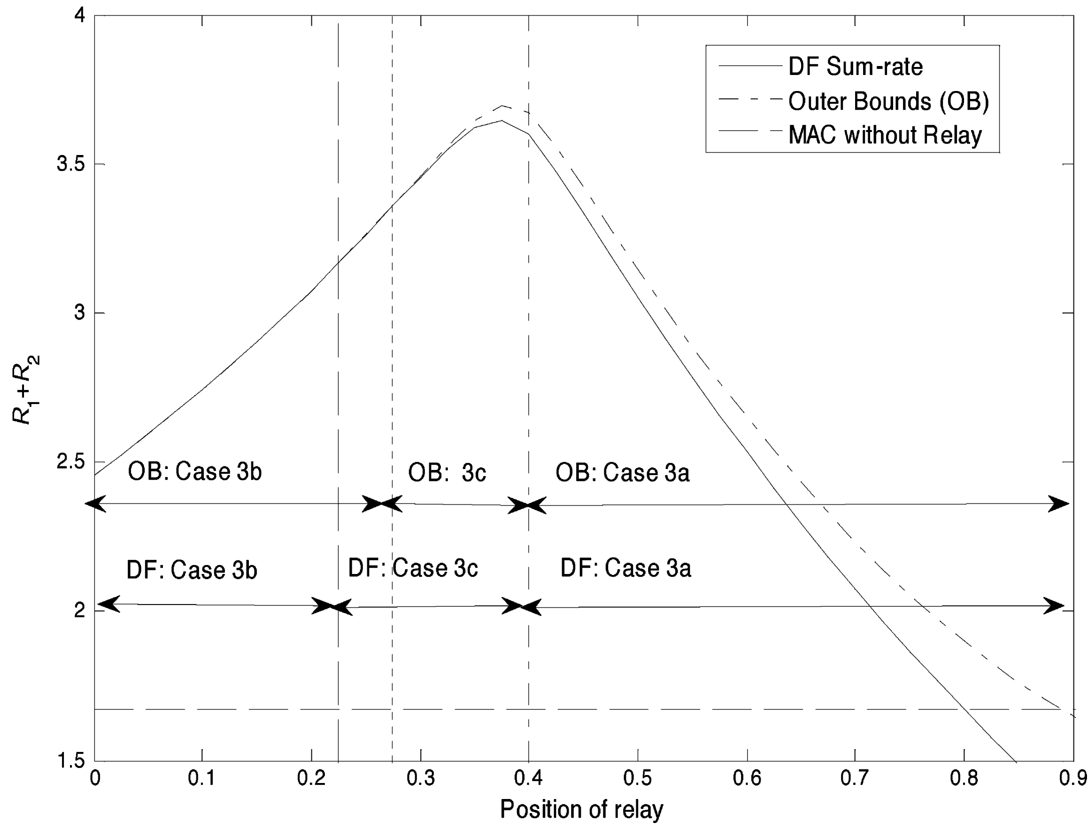

Fig. 8. Plot of inner and outer bounds on the sum-rate versus the relay position.

may be to use a combination of decoding and nondecoding strategies at the relay such as DF with compress-and-forward and amplify-and-forward. In these cases, optimizing the bandwidth parameter factor θ may also help. These may potentially enable better use of the relay resources albeit with increased complexity. Similar observations on combining strategies have also been made in [21] and [23].

3) Our results indicate that the channel model (orthogonal half-duplex model and choice of source-relay bandwidth fractions) as well as the network geometry determine the optimal solutions. In [21] and [23], the users and relay(s) are assumed to use orthogonal resources to avoid interference issues. We consider a model in which the users access the channel simultaneously over the same bandwidth but do not interfere with the relay. In all these models, the orthogonal use of bandwidth resources limits performance but allows practical implementation. However, the multiple-access nature of our model allows interference. In fact, while the sum-rate optimal DF power policies for our model involve opportunistic scheduling of users, which suggests scheduling users to transmit using orthogonal resources, our results clearly demonstrate the effect of the bandwidth fraction and the network geometry in determining the appropriate receiver to which to schedule all or a subset of users and highlight the subset of users for which the relay is most useful. These results can, therefore, potentially enable better user scheduling and bandwidth allocation in general multiple-access relay networks such as those studied in [21] and [23].

4) *Effect of user and relay powers*: While not explicitly shown in our illustration, we briefly remark on the effect of user and relay power on the DF sum-rate for a given geometry. First we note that for fixed user and relay average transmit powers, changing the value of θ is equivalent to changing the transmit powers in a specific way, i.e., increasing $\bar{P}_k$ by $1/\theta$, $k = 1, 2$, and $\bar{P}_r$ by $1/\bar{\theta}$. Our illustration demonstrates that for the range of θ,, and hence, the resulting small range of average node powers, and for the specific geometries considered, the optimizing case remains unchanged. In general, however, for a given geometry, the case maximizing the DF sum-rate will change when the transmit power of any node changes. For special cases such as symmetric or skewed geometries studied here, if the user powers scale proportionately (for fixed relay power and θ), then the optimizing case will remain unchanged.

VIII. Concluding Remarks

We have developed the maximum DF sum-rate and the sum-rate optimal power policies for an ergodic fading K-user half-duplex Gaussian MARC. The MARC is an example of a multiterminal network for which the multidimensionality of the policy set, the signal space, and the network topology space contribute to the complexity of developing capacity results resulting in few, if any, design rules for real-world communication networks. For a DF relay, the polymatroid intersection lemma we presented here allowed us to simplify the otherwise complicated analysis of developing the DF sum-rate optimal power policies for the two-user and K-user orthogonal MARC

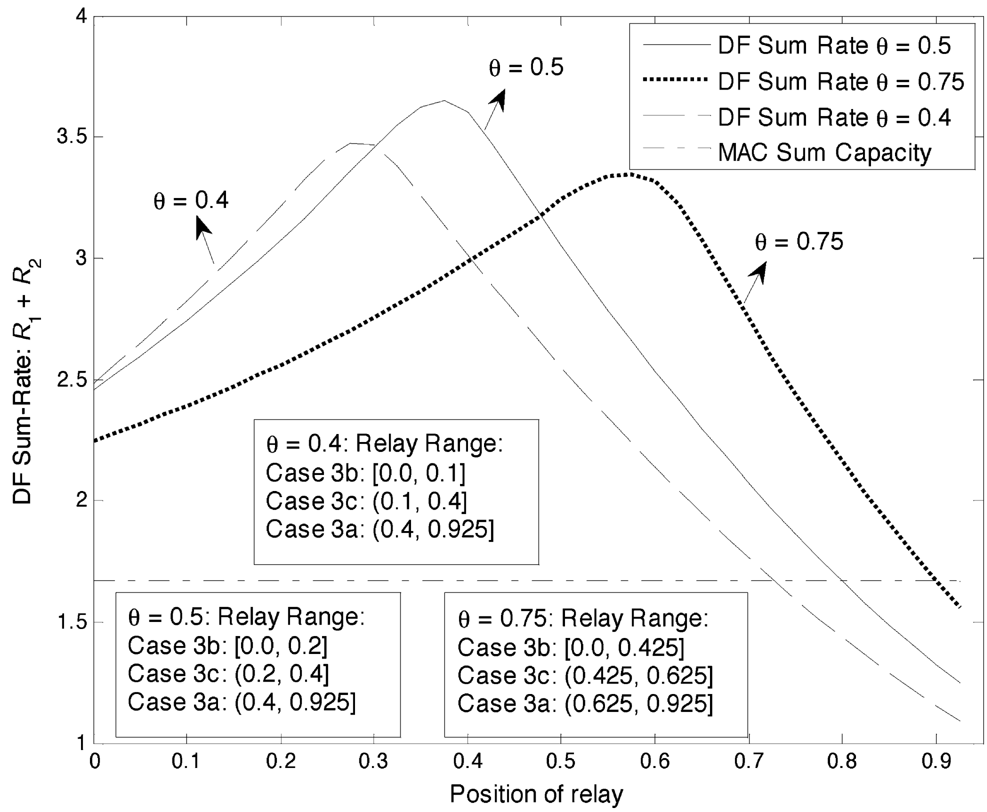

Fig. 9. DF sum rates achieved for $\theta = 0.4$, 0.5, and 0.75 for Topology 2.

and the K-user outer bounds. The lemma allowed us to develop a broad topological classification of fading MARCs into one of following three types:

i) *partially clustered MARCs* in which a subset of all users form a cluster with the relay while the complementary subset of users form a cluster with the destination;

ii) *clustered MARCs* comprised of either sources-relay or relay-destination clustered networks;

iii) *arbitrarily-clustered MARCs* that are a combination of either the two clustered models or of a clustered and a partially clustered model.

The optimal policies for the inner DF and the outer cutset bounds for the orthogonal MARC model studied here lead to the following observations:

- DF achieves the sum-capacity of a class of *source-relay clustered orthogonal MARCs* for which the combined link from all sources and the relay to the destination, i.e., the link achieving the K-user sum-rate at the destination is the bottle-neck link. Furthermore, DF achieves the capacity region when for every weighted sum of user rates, the limiting bound is the weighted rate-sum achieved at the destination.
- For this sum-capacity achieving case, the optimal user policies for both the orthogonal and nonorthogonal MARCs are multi-user opportunistic water-filling solutions over their links to the destination and the optimal relay policy is a water-filling solution over its direct link to the destination.
- And for the remaining classes of MARCs, the optimal users policies are water-filling and nonwater-filling solutions for the partially clustered and arbitrarily clustered models, respectively.

For the partially clustered cases, we have shown that the optimal policy for each user is multi-user water-filling over its bottle-neck link to one of the receivers. Thus, the users that are clustered with the destination are forced to transmit at a lower rate to allow decoding of their signals at the relatively distant relay. Our results suggest that a useful practical strategy for the partially clustered topologies may be to allow those distant users that present little interference at the relay to communicate directly with the destination.

The optimal relaying strategy for all except the capacity achieving clustered case described above remains open. Given the complexity of finding the optimal signaling schemes for a given performance metric in multiterminal networks, a natural extension to this work could be to understand the gap in spectral efficiency between DF and the cutset outer bounds for fading MARCs using layered deterministic models. Such bounds have been developed recently for time-invariant interference channels and relay channels in [43] and [44], respectively, for fading Gaussian broadcast channels with no channel state information at the transmitter in [45], and for fading interference channels in [46] and [47].

A note on complexity: our theoretical analysis distinguishes between all possible polymatroid intersection cases in determining the optimal policy for a K-user system and, therefore, has a complexity that grows exponentially in the number of users. In practice, however, for two intersecting polymatroids the maximum of a weighted sum of rates and the optimizing policies can be computed using *strongly polynomial-time* algorithms [24, Theorem 47.4]. Finally, our analysis can also be extended to develop the DF regions for a more general nonorthogonal half-duplex model as well as the full-duplex MARC model. One can also study the ergodic rate regions for other relaying schemes studied for the MARC such as compress-and-forward and partial decode-and-forward [10], [48].

Appendix
Proof of Theorem 2

The sum-rate maximizing DF power policy $\underline{P}^*(\underline{H})$ in Theorem 2 is obtained by sequentially determining the power policies $\underline{P}^{(i)}(\underline{H})$ and $\underline{P}^{(l,n)}(\underline{H})$ that maximize the sum-rate for cases i and (l,n), respectively, over all $\underline{P}(\underline{H}) \in \mathcal{P}$, until one of them satisfies the conditions for its case. The Lagragian maximizing the sum-rate for any case is given by

$$\mathcal{L}^{(\cdot)} = S_\theta^{(\cdot)} - \sum_{k\in\mathcal{T}} \nu_k \mathbb{E}[P_k(\underline{H}) - \bar{P}_k] + \sum_{k\in\mathcal{T}} \lambda_k P_k(\underline{H}) \quad (73)$$

where, for all $k \in \mathcal{T}, \nu_k$ are the dual variables associated with the power constraints in (5), $\lambda_k \geq 0$ are the dual variables associated with the positivity constraints $P_k(\underline{H}) \geq 0$, and $S_\theta^{(\cdot)}$ is given by (14) for each case. We now detail the optimization for each case separately starting with case 1.

Case 1: The optimal policy is $\underline{P}^{(1)}(H)$ that maximizes (73) if it belongs to the open set $\mathcal{B}_1$ defined by the conditions

$$R_{\{1\},d}\left(\underline{P}^{(1)}(\underline{H})\right) < R^{\min}_{\{1\},r}\left(\underline{P}^{(1)}(\underline{H})\right) \text{ and} \quad (74a)$$

$$R_{\{2\},r}\left(\underline{P}^{(1)}(\underline{H})\right) < R^{\min}_{\{2\},d}\left(\underline{P}^{(1)}(\underline{H})\right) \quad (74b)$$

where

$$\begin{aligned} R^{\min}_{1,r}(\underline{P}(\underline{H})) &= \theta I(X_1; Y_r \,|\, \underline{H}) \\ &= \mathbb{E}\left[\theta C\left(\frac{|H_{r,1}|^2 P_1(\underline{H})/\theta}{1+|H_{r,2}|^2 P_2(\underline{H})/\theta}\right)\right] \quad (75) \\ R^{\min}_{2,d}(\underline{P}(\underline{H})) &= \theta I\left(X_2; Y_d|\underline{H}\right) \\ &= \mathbb{E}\left[\theta C\left(\frac{|H_{d,2}|^2 P_2(\underline{H})/\theta}{1+|H_{d,1}|^2 P_1(\underline{H})/\theta}\right)\right]. \quad (76) \end{aligned}$$

The KKT conditions for this case simplify to (17) with the superscript set to 1. One can verify in a straightforward manner that these KKT conditions result in the power policies $P_k^{(1)}(\underline{h})$ and $P_r^{(1)}(\underline{h})$ given by (20) and (21), respectively.

Case 2: The optimal policy $\underline{P}^{(2)}(\underline{H})$ maximizes (73) if it belongs to the open set $\mathcal{B}_2$ given by the conditions

$$R_{\{1\},r}(\underline{P}(\underline{H})) < R^{\min}_{\{1\},d}(\underline{P}(\underline{H})) \quad \text{and} \quad (77a)$$

$$R_{\{2\},d}(\underline{P}(\underline{H})) < R^{\min}_{\{2\},r}(\underline{P}(\underline{H})) \quad (77b)$$

where $R^{\min}_{\{2\},r}$ and $R^{\min}_{\{1\},d}$ are given by (75) and (76), respectively, after replacing the user indices 1 by 2 and 2 by 1. The optimal $P_k^{(2)}(\underline{H})$ and $P_r^{(2)}(\underline{H})$ are given by (20) and (21), respectively, with $(k,m) = (1,r),(2,d)$ provided $\underline{P}^{(2)}(\underline{H})$ satisfies (77).

Case 3: Consider the three cases $3a, 3b$, and $3c$ shown in Fig. 3. Substituting the appropriate $S^{(i)}, i = 3a, 3b, 3c$, in (73), the sum-rate optimization for all three cases can be written as

$$\max_{\underline{P}} \min(R_{\mathcal{K},r}, R_{\mathcal{K},d}) \quad (78)$$

subject to average power and positivity constraints on P_k for all k, where $R_{\mathcal{K},j}$ denotes the sum-rate bound at receiver $j, j = d, r$. We write $\mathcal{B}_3$ to denote the open set consisting of all $\underline{P}(\underline{H}) \in \mathcal{P}$ that do not satisfy (74) and (77) either as strict inequalities or as a mixture of equalities and inequalities, where by a mixture we mean that a subset of the inequalities in (74) and (77) are satisfied with equality. We will later show that such sets of mixed equalities and inequalities in (74) and (77) corresponds to conditions for the various boundary cases (see also Figs. 4 and 5). Thus, $\underline{P}(\underline{H}) \in \mathcal{B}_3$ only when it does not satisfy the conditions for the inactive and the active-inactive boundary cases. By definition, $\mathcal{B}_3 = \mathcal{B}_{3a} \cup \mathcal{B}_{3b} \cup \mathcal{B}_{3c}$, where $\mathcal{B}_i, i = 3a, 3b, 3c$, is defined for case i below.

The optimization in (78) is a multi-user generalization of the single-user *max-min* problem studied in [6] for the orthogonal single-user relay channel. The classical results on minimax optimization also apply to the multi-user sum-rate optimization in (78), and thus, the optimal policy $\underline{P}^{(i)}(\underline{H}), i = 3a, 3b, 3c$, satisfies one of following three conditions:

$$\begin{aligned} \text{Case } 3a: \quad & R_{\mathcal{K},r}|_{\underline{P}^{(3a)}(\underline{H})} < R_{\mathcal{K},d}|_{\underline{P}^{(3a)}(\underline{H})} \\ & R^{\min}_{\{1\},r}|_{\underline{P}^{(3a)}(\underline{H})} < R_{\{1\},d}|_{\underline{P}^{(3a)}(\underline{H})} \\ & R^{\min}_{\{2\},r}|_{\underline{P}^{(3a)}(\underline{H})} < R_{\{2\},d}|_{\underline{P}^{(3a)}(\underline{H})} \quad (79) \\ \text{Case } 3b: \quad & R_{\mathcal{K},r}|_{\underline{P}^{(3b)}(\underline{H})} > R_{\mathcal{K},d}|_{\underline{P}^{(3b)}(\underline{H})} \\ & R^{\min}_{\{2\},d}|_{\underline{P}^{(3b)}(\underline{H})} < R_{\{2\},r}|_{\underline{P}^{(3b)}(\underline{H})} \\ & R^{\min}_{\{1\},d}|_{\underline{P}^{(3b)}(\underline{H})} > R_{\{1\},r}|_{\underline{P}^{(3b)}(\underline{H})} \quad (80) \\ \text{Case } 3c: \quad & R^{\min}_{\{2\},d}|_{\underline{P}^{(3c)}(\underline{H})} < R_{\{2\},r}|_{\underline{P}^{(3c)}(\underline{H})} \\ & R^{\min}_{\{1\},d}|_{\underline{P}^{(3c)}(\underline{H})} > R_{\{1\},r}|_{\underline{P}^{(3c)}(\underline{H})}. \quad (81) \end{aligned}$$

Note that the conditions in (79)–(81), evaluated at any $P \in \mathcal{B}_3$, are also conditions defining the sets $\mathcal{B}_{3a}, \mathcal{B}_{3b}$, and $\mathcal{B}_{3c}$, respectively. We now present the optimal policies and sum-rates for each case in detail.

Case 3a: For this case, from the KKT conditions in (17), $f^{(3a)}$ in (23) depend only the sum-rate and channels gains of the two users at the relay. Thus, the problem simplifies to that for a MAC at the relay and the classical multi-user water-filling solution developed in [25] and [26] applies. The optimal user policies are thus given by (24), in which with the exception of the equality condition in (24), the optimal policies are unique, i.e., the optimal $P_k^{(3a)}(\underline{H})$ at user k in (24) is an opportunistic water-filling solution that exploits the fading diversity in a multi-access channel from the sources to the relay. If the channel gains are continuously distributed, the equality condition occurs with probability 0. Furthermore, even if the distributions were not continuous, one could choose to schedule one user or the other when the equality condition is met, thereby maintaining the opportunistic allocation policy. Finally, the optimal power policy at the relay is not explicitly obtained from the Lagrangian for this case as $S_\theta^{(3a)}$ is the sum-rate achieved by the sources at the relay. However, since the sum-rate at the relay for this case is smaller than that at the destination, choosing the water-filling policy at the relay that maximizes the relay-destination link preserves the condition for this case, and thus, $P_r^{(3a)}(\underline{H})$ is given by (21). When $P^{(3a)}(\underline{H}) \in \mathcal{B}_3$, the requirement of satisfying (79), i.e., $\underline{P}^{(3a)}(\underline{H}) \in \mathcal{B}_{3a}$, simplifies to a threshold condition $\bar{P}_r > P_u(\bar{P}_1, \bar{P}_2)$ where $\bar{P}_k, k \in \mathcal{T}$, is defined in (5) and the threshold $P_u(\bar{P}_1, \bar{P}_2)$ is obtained by setting (79) to an equality. When $\underline{P}^{(3a)}(\underline{H}) \in \mathcal{B}_3$ but $\underline{P}^{(3a)}(\underline{H}) \notin \mathcal{B}_{3a}, R_1 + R_2$ is maximized by either *case 3b* or *case 3c*. For $\underline{P}^{(3a)}(\underline{H}) \notin \mathcal{B}_3$, as argued in Section IV, the sum-rate is not maximized by any $\underline{P}(\underline{H}) \in \mathcal{B}_3$.

Case 3b: The optimal policy $P_k^{(3b)}(\underline{H})$ at user k for this case satisfies the KKT conditions in (17) with $f_k^{(i)} = f_k^{(3b)}$. As with case $3a$, here too, the optimal policy is an opportunistic water-filling solution and is given by (24) with the subscript 'r' changed to 'd' for all k and with the superscript $i = 3b$. Further, for the relay node, the optimal $P_r^{(3b)}(H)$ satisfies the KKT conditions in (17) for $f_r^{(3b)} = f_r^{(1)}$, and is given by the water-filling solution in (21). Finally, for $\underline{P}^{(3b)}(\underline{H}) \in \mathcal{B}_3$, the requirement $\underline{P}^{(3b)}(\underline{H}) \in \mathcal{B}_{3b}$ simplifies to satisfying the threshold condition $\bar{P}_r < P_l(\bar{P}_1, \bar{P}_2)$ where $P_l(\bar{P}_1, \bar{P}_2)$ is determined by setting (80) to an equality.

Case 3c (*equal-rate policy*): The function $f_k^{(3c)}$ is a weighted sum of $f_k^{(3a)}$ and $f_k^{(3b)}$ in which the Lagrange multiplier α accounts for the boundary condition in (81). Substituting $f_k^{(3c)}$ in (17), we have the following KKT conditions:

$$\frac{\alpha|h_{r,k}|^2}{1+\sum_{k=1}^{2}|h_{r,k}|^2\frac{P_k(\underline{h})}{\theta}} + \frac{(1-\alpha)|h_{d,k}|^2}{1+\sum_{k=1}^{2}|h_{d,k}|^2\frac{P_k(\underline{h})}{\theta}} \le \nu_k \ln 2 \quad \text{with equality for } P_k(\underline{h}) > 0, k=1,2 \tag{82}$$

from which the optimal power policies simplify to (30). Determining the optimal $P_k^{(3c)}(\underline{h}), k = 1, 2$, requires verifying each one of the three conditions in (30). Note that in contrast to cases $3a$ and $3b$, the opportunistic scheduling policy in (30) also depends on the user policies in addition to the channel states. Furthermore, the optimal solutions $P_k^{(3c)}(\underline{H})$ do not take a water-filling form. Thus, for a given $P_1(\underline{h}), P_2(\underline{h})$ is given by

$$P_2(\underline{h}) = \text{positive root } x \text{ of (84) if it exists, otherwise } 0 \tag{83}$$

where the root x is determined by the following equation:

$$\frac{\alpha\,|h_{r,2}|^2}{1+|h_{r,1}|^2\,\frac{P_1(\underline{h})}{\theta}+|h_{r,2}|^2\,\frac{x}{\theta}} + \frac{(1-\alpha)|h_{d,2}|^2}{1+|h_{d,k}|^2\frac{P_1(\underline{h})}{\theta}+|h_{d,k}|^2\,\frac{x}{\theta}} = \nu_2 \ln 2. \tag{84}$$

Using $P_2(\underline{h})$ given by (84), $P_1(\underline{h})$ is obtained as the root of

$$\frac{\alpha|h_{r,1}|^2}{1+|h_{r,1}|^2\frac{P_1(\underline{h})}{\theta}+|h_{r,2}|^2\frac{P_2(\underline{h})}{\theta}} + \frac{(1-\alpha)|h_{d,1}|^2}{1+|h_{d,k}|^2\frac{P_1(\underline{h})}{\theta}+|h_{d,k}|^2\frac{P_2(\underline{h})}{\theta}} = \nu_1 \ln 2. \tag{85}$$

Thus, for all $\underline{h}$, starting with an initial $P_1(\underline{h})$, we iteratively obtain $P_1(\underline{h})$ and $P_2(\underline{h})$ until they converge to $P_1^{(3c)}(\underline{h})$ and $P_2^{(3c)}(\underline{h})$. The proof of convergence is detailed below. Finally, the optimal policies are determined over all $\alpha \in [0, 1]$ to find an α^* that satisfies the equal rate condition in (81).

Proof of Convergence: The proof follows along the same lines as that detailed in [42, p. 3440] and relies on the fact that the maximizing function $S_\theta^{(3c)}$ is a strictly concave function of $P_1(\underline{H})$ and $P_2(\underline{H})$ and is bounded from above because of the power constraints at the source and relay nodes. At each iteration, the optimal $P_1(\underline{H})$ and $P_2(\underline{H})$ are the KKT solutions that maximize the objective function. Thus, after each iteration, the objective function either increases or remains the same. It is easy to check that for a given $P_1(\underline{H})$ the objective function is a strictly concave function of $P_2(\underline{H})$, and thus, (84) yields a unique value of $P_2(\underline{H})$. Furthermore, the objective function is also a strictly concave function of $P_1(\underline{H})$ for a fixed $P_2(\underline{H})$. Thus, as the objective function converges, $(P_1(\underline{H}), P_2(\underline{H}))$ also converges. Finally, $P_1(\underline{H})$ and $P_2(\underline{H})$ converge to the solutions of the KKT conditions, which is sufficient for $(P_1(\underline{H}), P_2(\underline{H}))$ to be optimal since the objective function is concave over all $\underline{P}(\underline{H}) \in \mathcal{P}$.

Case 4: (Boundary Cases): Recall that we define the sets $B_i, i = 1, 2, 3a, 3b, 3c$, as open sets to ensure that an optimal $\underline{P}^{DF}$ maximizes the sum-rate for a case only if it satisfies the conditions for that case. Since an optimal policy can lie on the boundary of any two such cases, we also consider six additional cases each of which lies at the boundary of an inactive and an active case. These boundary cases result when the conditions for an inactive case $l, l = 1, 2$, and an active case $n, n = 3a, 3b, 3c$, are such that the sum-rate is the same for both cases. We consider each of the six boundary cases separately and develop the optimal $\underline{P}^{(l,n)}(\underline{H})$ for each case. The requirement that the optimal $\underline{P}^{(l,n)}(\underline{H})$ satisfies the condition $S_\theta^{(l)} = S_\theta^{(n)}$ for the boundary case (l, n) simplifies to

$$\text{case } (1,3a): R_{\{1\},d} + R_{\{2\},r} = R_{\mathcal{K},r} < R_{\mathcal{K},d} \tag{86}$$
$$\text{case } (1,3b): R_{\{1\},d} + R_{\{2\},r} = R_{\mathcal{K},d} < R_{\mathcal{K},r} \tag{87}$$
$$\text{case } (1,3c): R_{\{1\},d} + R_{\{2\},r} = R_{\mathcal{K},r} = R_{\mathcal{K},d} \tag{88}$$
$$\text{case } (2,3a): R_{\{1\},r} + R_{\{2\},d} = R_{\mathcal{K},r} < R_{\mathcal{K},d} \tag{89}$$
$$\text{case } (2,3b): R_{\{1\},r} + R_{\{2\},d} = R_{\mathcal{K},d} < R_{\mathcal{K},r} \tag{90}$$
$$\text{case } (2,3c): R_{\{1\},r} + R_{\{2\},d} = R_{\mathcal{K},d} = R_{\mathcal{K},r} \tag{91}$$

where the conditions in (86)–(91) are evaluated at the appropriate $\underline{P}^{(l,n)}(\underline{H})$. In addition, to ensure disjoint sets, from (53) we require that $\underline{P}^{(l,n)}(\underline{H})$ also satisfy (45)–(50) which are the conditions defining the the sets $\mathcal{B}_{(1,3a)}$ through $\mathcal{B}_{(2,3c)}$, respectively. Using (45)–(50), we write the Lagrangian for all boundary cases except cases $(1, 3c)$ and $(2, 3c)$ as

$$\mathcal{L}^{(l,n)} = \alpha S_\theta^{(l)} + (1-\alpha) S_\theta^{(n)} - \sum_{k\in\mathcal{T}} \nu_k \mathbb{E}[P_k(\underline{H}) - \bar{P}_k] + \sum_{k\in\mathcal{T}} \lambda_k P_k(\underline{h}), \quad l = 1,2, n = 3a, 3b \tag{92}$$

$$\lambda_k P_k(\underline{h}) \ge 0, \quad \text{for all } \underline{h} \text{ and } k \in \mathcal{T} \tag{93}$$

and the Lagrangian for cases $(1,3c)$ and $(2,3c)$ as

$$\mathcal{L}^{(l,3c)} = \alpha_1 S_\theta^{(l)} + \alpha_2 S_\theta^{(3a)} + (1-\alpha_1-\alpha_2) S_\theta^{(3b)} - \sum_{k\in\mathcal{T}} \nu_k \mathbb{E}[P_k(\underline{H}) - \bar{P}_k] + \sum_{k\in\mathcal{T}} \lambda_k P_k(\underline{h}) \quad l=1,2 \tag{94}$$

$$\lambda_k P_k(\underline{h}) \geq 0, \quad \text{for all } \underline{h} \text{ and } k \in \mathcal{T} \tag{95}$$

where ν_k and $\lambda_k \geq 0$ are dual variables associated with the average power and positivity constraints on P_k, respectively. The variable α is the dual variable associated with all boundary cases with a single boundary condition while α_1 and α_2 are the dual variables associated with cases $(1,3c)$ and $(2,3c)$. The resulting KKT conditions, one for each $P_k(\underline{h}), k=1,2,r$, are

$$\text{Case } (l,n\neq 3c): \frac{\partial \mathcal{L}^{(l,n)}}{\partial P_k(\underline{h})} = f_k^{(l,n)} = \alpha f_k^{(l)} + (1-\alpha) f_k^{(n)} \leq \nu_k \ln 2 \tag{96}$$

$$\text{Case } (l,n=3c): \frac{\partial \mathcal{L}^{(l,n)}}{\partial P_k(\underline{h})} = f_k^{(l,n)} = \alpha_1 f_k^{(l)} + \alpha_2 f_k^{(3a)} + (1-\alpha_1-\alpha_2) f_k^{(3b)} \leq \nu_k \ln 2 \tag{97}$$

where $f_k^{(l)}$ and $f_k^{(n)}$ are as defined earlier for cases l and n and equality holds in (96) and (97) for $P_k(\underline{h}) > 0$, for all $\underline{h}$. We now present the optimal policies for each case separately.

Case $(1,3a)$: From (96), the KKT conditions for this case are

$$f_1^{(1,3a)} = \frac{\alpha|h_{d,1}|^2}{1+|h_{d,1}|^2 P_1(\underline{h})/\theta} + \frac{(1-\alpha)|h_{r,1}|^2}{1+\sum_{j=1}^2 |h_{r,j}|^2 P_j(\underline{h})/\theta} \leq \nu_1 \ln 2, \text{with equality if } P_1(\underline{h}) > 0 \tag{98}$$

$$f_2^{(1,3a)} = \frac{\alpha|h_{r,2}|^2}{1+|h_{r,2}|^2 P_2(\underline{h})/\theta} + \frac{(1-\alpha)|h_{r,2}|^2}{1+\sum_{j=1}^2 |h_{r,j}|^2 P_j(\underline{h})/\theta} \leq \nu_2 \ln 2, \text{with equality if } P_2(\underline{h}) > 0 \tag{99}$$

$$f_r^{(1,3a)} = \frac{\alpha|h_{d,r}|^2}{1+|h_{d,r}|^2 P_r(\underline{h})/\bar{\theta}} \leq \nu_r \ln 2 \quad \text{with equality if } P_r(\underline{h}) > 0 \tag{100}$$

which results in (39). As in case $3c$, the optimal policies take an opportunistic nonwater-filling form and in fact can be obtained by the iterative algorithm described for that case. Finally, from (100), the optimal $P_r^{(1,3a)}(\underline{H})$ is given by (32).

Case $(1,3b)$: The analysis for this case mirrors that for case $(1,3a)$ and the optimal user policies are opportunistic nonwater-filling solutions given by (39) with $f_k^{(3a)}$ replaced by $f_k^{(3b)}, k=1,2$. On the other hand in contrast to case $(1,3a)$ where $f_r^{(3a)} = 0$, since both $f_r^{(1)}$ and $f_r^{(3b)}$ are nonzero, the optimal relay policy $P_r^{(2,3a)} = P_r^{(1)}$.

Case $(1,3c)$: For this case, the KKT conditions in (97) involves a weighted sum of $f_k^{(l)}, f_k^{(3a)}$, and $f_k^{(3b)}$. Thus, for $k=1,2,(k,m)=(1,d),(2,r)$, we have the KKT conditions

$$f_1^{(1,3c)} = \alpha_1 f_1^{(1)} + \alpha_2 f_1^{(3a)} + (1-\alpha_1-\alpha_2) f_1^{(3b)} \leq \nu_1 \ln 2, \text{with equality if } P_1(\underline{h}) > 0 \tag{101}$$

$$f_2^{(1,3c)} = \alpha_1 f_2^{(1)} + \alpha_2 f_2^{(3a)} + (1-\beta)(1-\alpha) f_2^{(3b)} \leq \nu_2 \ln 2, \text{with equality if } P_2(\underline{h}) > 0 \tag{102}$$

$$f_r^{(1,3c)} = (1-\alpha_2)|h_{d,r}|^2 / C(|h_{d,r}|^2 P_r/\bar{\theta}) \leq \nu_r \ln 2, \text{with equality if } P_r(\underline{h}) > 0 \tag{103}$$

where α_1 and α_2 are the dual variables associated with the equalities $R_{\mathcal{K},d} = R_{\{1\},d} + R_{\{2\},r}$ and $R_{\mathcal{K},d} = R_{\mathcal{K},r}$, respectively, in (88). From (101) and (102), one can verify that the optimal user policies are opportunistic nonwater-filling solutions given by (39) with the superscript $(1,3a)$ replaced by $(1,3c)$. Finally, $P_r^{(1,3c)}(\underline{H})$ is given by the water-filling solution in (32) with α replaced by $(1-\alpha_2)$.

Case $(2,3a)$: The optimal user policies for this case and the KKT conditions they satisfy are given by (98), (99), and (39) when $f_k^{(1)}$ is replaced by $f^{(2)}$, for all k, and $g_k^{(\cdot)}$ is superscripted by $(2,3a)$. Thus, here too, the optimal user policies are opportunistic nonwater-filling solutions. The optimal relay policy $P_r^{(2,3a)}(\underline{H})$ is the same as that obtained in case $(1,3a)$.

Case $(2,3b)$: The optimal user policies $P_k^{(2,3b)}(\underline{H}), k=1,2$, are again opportunistic nonwater-filling solutions and are given by (98), (99), and (39) when $f_k^{(1)}$ and $f_k^{(3a)}$ are replaced by $f^{(2)}$ and $f^{(3b)}$, respectively, for all k, and $g_k^{(\cdot)}$ is superscripted by $(2,3b)$. The optimal relay policy $P_r^{(2,3b)}(\underline{H})$ is the same as that for case $(1,3b)$.

Case $(2,3c)$: The optimal policy vector $\underline{P}^{(2,3c)}(\underline{H})$ is the same as that for *case* $(1,3c)$ with $f_k^{(1)}$ is replaced by $f^{(2)}$, for all k, and with the superscript $(2,3c)$.

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